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Abstract: In this work, we study the exponential stability of the stationary distribution of a McKean-Vlasov equation, of nonlinear hyperbolic type which was recently derived in [8, 18]. We complement the convergence result proved in [18] using tools from dynamical systems theory. Our proof relies on two principal arguments in addition to a Picard-like iteration method. First, the linearized semigroup is positive which allows to precisely pinpoint the spectrum of the infinitesimal generator. Second, we use a time rescaling argument to transform the original quasilinear equation into another one for which the nonlinear flow is differentiable. Interestingly, this convergence result can be interpreted as the existence of a locally exponentially attracting center manifold for a hyperbolic equation.

Key-words: McKean-Vlasov equations, nonlocal nonlinear transport equation, boundary condition, stationary distribution, nonlinear stability, center manifold.

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Stabilité exponentielle de la distribution stationnaire d'un champ moyen de réseau de neurones à spikes.

Résumé : Dans ce travail, nous étudions la stabilité exponentielle de la distribution stationnaire d'une équation de McKean-Vlasov, de type hyperbolique non linéaire qui a été récemment introduite dans [8, 18]. Nous complètons le résultat de convergence prouvé dans [18] en utilisant des outils de la théorie des systèmes dynamiques. Notre preuve repose sur deux arguments principaux en plus d'une méthode d'itération de type Picard. Premièrement, le semigroupe linéarisé est positif, ce qui permet de localiser avec précision le spectre du générateur infinitésimal. Deuxièmement, nous utilisons un changement de variable en temps pour transformer l'équation quasilinéaire originale en une autre pour laquelle le flot non linéaire est différentiable. Il est intéressant de noter que ce résultat de convergence peut être interprété comme l'existence d'un variété central localement exponentiellement attractive pour une équation hyperbolique.

Mots-clés : équation de McKean–Vlasov, distribution stationnaire, équation de transport nonlinéaire et non locale, stabilité non linéaire, variété centrale

Contents

1 Introduction

In [8, 18], the authors derived mean-field equations for a network of excitatory spiking neurons in the limit of a large number of neurons (see also [36]). It is based on a recently published model of simple neural network [8] in which the spiking dynamics of the individual neurons is modeled with a jump process rather than with threshold crossing [19] or blow up of the membrane potential [26]. The distribution $x \to g(t, x)$ of the membrane potential of the limiting mean-field process solves:

$$\begin{cases} \frac{\partial}{\partial t} g(t,x) = \left[\lambda x - \int_0^\infty (f(v) + \lambda v)\, g(t,v) dv\right] \partial_x g(t,x) + [\lambda - f(x)]\, g(t,x), \quad t, x > 0 \\ g(t,0) = \dfrac{\int_0^\infty fg}{\int_0^\infty (f(v) + \lambda v)\, g(t,dv)} \\ g(0,\cdot) \in L^1_+(\mathbb{R}_+) \end{cases}$$

where f, is the rate function which is positive on $\mathbb{R}_{>0}$. In addition to the derivation of the mean-field equations, the authors of [18] computed an analytical formula for the stationary distribution of the equations. In the case $\lambda = 0$, they were able to prove that

$$\|g(t) - g_\infty\|_{\mathrm{L}^1} \overset{t\to\infty}{\to} 0$$

where g_∞ is the unique stationary distribution of the system, note that it has a density. The above limit holds for some regular enough initial conditions. In the case where $f(x) \geq cx^\xi$ for all $x \in [0, 1]$ with $c > 0, \xi \geq 1$, they showed that the above convergence is $O\left((1 + t)^{-1/\xi}\right)$.

The main focus of the present work is the case $\lambda = 0$. Indeed, in [14], we provided numerical evidences for oscillatory patterns when $\lambda > 0$ thereby suggesting that the above convergence result is not true for all $\lambda > 0$. The advantage of the case $\lambda = 0$ is that it removes the pre-factor λx which allows to use a time rescaling to avoid studying a quasilinear equation [32] and to build a differentiable nonlinear semigroup of solutions. Finally, it also removes the boundary condition. The equation thus reads:

$$\begin{cases} \partial_t g(t,x) = -\left(\int_0^\infty f(v) g(t,v) dv\right) \partial_x g(t,x) - f(x) g(t,x), \; x, t > 0 \\ g(t,0) = 1, \\ g(0,\cdot) = g_0 \in L^1_+(\mathbb{R}_+). \end{cases} \tag{1}$$

In this work, we revisit the convergence to the stationary distribution from a dynamical systems point of view in order to prove that the convergence is locally exponential in time.

Note that there is a one dimensional family of stationary solutions $(g_\alpha)_{\alpha>0}$ and only one of them g_∞ is a stationary distribution *i.e.* with integral equal to one. This family is given by:

$$g_\alpha(x) = \exp\left(-\frac{1}{\alpha}\int_0^x f\right), \; \int_0^\infty f g_\alpha = \alpha > 0. \tag{2}$$

The existence of this family implies that zero is in the spectrum of the linearized equation: the principle of linearized stability does not apply. There are several strategies to prove the nonlinear stability of g_α in this case apart from entropy methods [33] which we have not looked at.

The first relies on the local attractiveness of a center manifold composed of the family $(g_\alpha)_{\alpha>0}$. Indeed, the analysis of the spectrum shows that the center manifold should be one dimensional. To prove nonlinear stability, one would need to prove that the center manifold is locally attracting [24, 39]. Unfortunately, it is difficult to achieve such program as it relies heavily on the fact that the linear flow must be regularizing, which in the case of transport equations, requires to use very regular initial conditions.

The second idea, which we shall rely on, starts with the observation that the flow of (1) conserves the mass. Hence, the nonlinear flow is foliated by the linear form $g \to \int_0^\infty g$. The dynamics on each hyperplane possesses a unique equilibrium which is now hyperbolic. Thus, one can hope proving nonlinear stability by simpler means in this case.

Using the second idea, we prove the existence of an exponentially attracting center manifold which is transverse to the hyperplanes associated with the linear form $g \to \int_0^\infty g$. This is noticeable as such general result is not known for transport equations and for quasilinear equations. It is for example well known for delay differential equations [23, 12, 38, 40] which are a kind of transport equation with a nonlinear boundary condition.

The type of equations considered here is well studied in the population dynamics literature [21, 34, 41, 1, 33] but a complete analogy with (1) would require to introduce unbounded birth / death rates of the species which is less studied for modeling reasons. Another noticeable difference lies in the fact that the equations are considered on a non compact domain here. In the neuroscience community, these equations stems from a recent surge to put on rigorous grounds [7, 3, 9] mean-field of networks of spiking neurons and more precisely of integrate-and-fire neurons [35, 30]. However, this last mean-field equation exhibits blow up unlike the one that we study here because the spiking mechanism of individual neurons is based, here, on a jump process instead of threshold crossing. Additionally, the mean-field of spiking neurons modeled after Hawkes processes have been recently investigated [6, 4, 5, 13]: the proof of the convergence of the particle system is simpler. The mean-field equation in this case (see also [31]) is a nonlinear age-structured equation akin to the one mentioned above in the population dynamics context. They have been recently studied from a dynamical systems point of view [42, 28].

The plan of the paper is as follows. In section 3, we first study the linearized equation around a stationary point g_α in the space $\mathrm{L}^1(\mathbb{R}_+)$. Then, in section 4, we transform the nonlinear equation by a time rescaling and a cut-off to induce a differentiable nonlinear semigroup of solutions. We then use a variant of Picard theorem to prove nonlinear stability for the rescaled equation. Finally, we conclude with the **main result** in section 4.5 concerning with the local exponential stability of the stationary solution g_∞. For convenience, we re-state this result here.

Theorem 1.1 *Grant Assumptions 1 and 2. The distribution g_∞ is locally exponentially stable for the flow of (1) in $\widehat{\mathcal{X}}_2^{\mathbf{A}}$, that is for all $\epsilon > 0$ small enough, there is a neighborhood $\mathcal{V}_\epsilon \subset \{\phi \in \mathcal{X},\ \phi'' \in \mathcal{X},\ f\phi' \in \mathcal{X},\ f^2\phi \in \mathcal{X},\ \phi(0) = \phi'(0) = 0,\ \int \phi = 0\}$ such that*

$$\exists C_\epsilon \geq 1 \quad \forall g_0 \in g_\infty + \mathcal{V}_\epsilon,\ \forall t \geq 0 \ \ \|g(t) - g_\infty\|_{\mathrm{L}^1} \leq \|g(t) - g_\infty\|_{\mathcal{X}_2^{\mathbf{A}}} \leq C_\epsilon e^{\left(s(\mathbf{A}_|)+\epsilon\right)t}\|\phi\|_{2,\mathbf{A}}$$

where $s(\mathbf{A}_|) < 0$.

2 Notations and assumptions

Whenever possible, we shall write $\mathbb{C}_{\leq a} = \{z \in \mathbb{C} \mid \Re z \leq a\}$ and similarly for $\mathbb{C}_{\geq \ldots}$. We use the notation $f \lesssim g$ when there exists a constant $C > 0$ independent of the parameters of interest such that $f \leq Cg$.

We denote by $L^1(\mathbb{R}_+, d\mu)$ the space of integrable functions from $\mathbb{R}_+$ to $\mathbb{C}$ for the measure μ, we then define $\mathcal{X} = L^1(\mathbb{R}_+, dl)$ where l is the Lebesgue measure. We further denote by $L^1_+(\mathbb{R}_+)$ the subspace of non-negative functions and by $\hat{\mathcal{X}} = \left\{\phi \in \mathcal{X} \mid \int_0^\infty \phi = 0\right\}$ the subspace of functions of zero integral. We also define the two following linear forms respectively on $L^1(f(x)dx)$ and $\mathcal{X}$:

$$a(\phi) = \int_0^\infty f\phi, \quad I(\phi) = \int_0^\infty \phi.$$

We write H the Heaviside function $H(x) = 1$ if $x \geq 0$ and 0 otherwise.

For a linear operator $\mathbf{A} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, we write $\ker(\mathbf{A})$ its kernel and $Ran(\mathbf{A})$ its range. The resolvent operator $\mathbf{R}(\mu, \mathbf{A})$ of a closed operator $\mathbf{A}$ is $\mathbf{R}(\mu, \mathbf{A}) = (\mu \mathrm{Id} - \mathbf{A})^{-1}$ for μ in the resolvent set $\rho(\mathbf{A})$ of $\mathbf{A}$. Finally, we write $\Sigma(\mathbf{A})$ the spectrum of $\mathbf{A}$ and $s(\mathbf{A}) \stackrel{def}{=} \sup\{\Re\lambda \ :\ \lambda \in \Sigma(\mathbf{A})\}$ the spectral bound. For a family of bounded operator $(\mathbf{T}(t))_{t\geq 0}$, we write the growth bound $\omega_0(\mathbf{T}) \stackrel{def}{=} \inf\{\omega \in \mathbb{R}\ :\ \exists M_\omega \geq 1 \text{ such that } \|\mathbf{T}(t)\|_{\mathcal{L}(\mathcal{X})} \leq M_\omega e^{\omega t},\ \forall t \geq 0\}$.

The multiplication operator is written $\mathbf{M}_f : \phi \to f\phi$. When $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space, $I \subset \mathbb{R}$ an interval and $\phi \in C^0(I, \mathcal{Y})$, we write the sup-norm on $\mathcal{Y}$ as $\|\phi\|_{C^0(I,\mathcal{Y})} = \sup_{t\in I} \|\phi(t,\cdot)\|_{\mathcal{Y}}$. The shorter notation $\|\phi\|_{C^0}$ can be used if the interval I and the Banach space $\mathcal{Y}$ are clearly determined. Keeping the same notations, we write $C^0_b([s,\infty), \mathcal{Y})$ the Banach space of continuous functions bounded on $[s,\infty)$ with respect to the sup-norm on $\mathcal{Y}$.

We introduce a notation concerning the notion of Sobolev space [17] which is used all along in this paper. For a closed operator $\mathbf{C}$ on the domains $D(\mathbf{C}^n)$ and $\lambda \in \rho(\mathbf{C})$, we introduce the norms $\|\cdot\|_{n,\mathbf{C},\lambda} \stackrel{def}{=} \|(\lambda \mathrm{Id} - \mathbf{C})^n \cdot\|$ and call $\mathcal{Y}^{\mathbf{C}}_0 \stackrel{def}{=} \mathcal{Y}$, $\mathcal{Y}^{\mathbf{C}}_n \stackrel{def}{=} (D(\mathbf{C}^n), \|\cdot\|_{n,\mathbf{C}})$ the Sobolev space of order n associated with $\mathbf{C}$. Note that for each fixed $n \in \mathbb{N}$, all the norms $\|\cdot\|_{n,\mathbf{C},\lambda}$ are equivalent for $\lambda \in \rho(\mathbf{C})$ and are therefore written $\|\cdot\|_{n,\mathbf{C}}$ if no confusion is possible.

Following [18], we make the following assumptions concerning the rate function f:

Assumption 1 *f is convex increasing, $f(0) = 0$, $f(x) > 0$ for all $x > 0$, $\lim_\infty f = \infty$ and $f \in C^2(\mathbb{R}_+)$. Further assume that f is convex and that* $\sup\limits_{x\geq 1} \frac{f'(x)}{f(x)} + \frac{f''(x)}{f'(x)} < \infty$.

Assumption 2 *f is such that $f'(0) = 0$.*

From [18], this implies the following properties:

Remark 1 *Grant Assumption 1, we have the following properties:*

(i) *There is $c > 0$ such that $f(x) \geq cx$ for all $x \geq 1$.*

(ii) *For all $A > 0$, there is $C_A > 0$ such that for all $x \geq 0$, $f(x + A) \leq C_A(1 + f(x))$.*

(iii) *There is $C > 0$ such that $f(x) \leq C \exp(Cx)$ for all $x \geq 0$.*

(iv) *f is super additive that is: for all $(x, y) \in \mathbb{R}^2_+$, $f(x + y) \geq f(x) + f(y)$.*

3 Linear analysis

Let us consider the unique [18] stationary point g_∞ of the family $(g_\alpha)_{\alpha>0}$ such that $\int_0^\infty g_\infty = 1$ and define the stationary firing rate $a_\infty \stackrel{def}{=} \int_0^\infty f g_\infty$. We obtain $g_\infty(x) = \exp\left(-\frac{1}{a_\infty}\int_0^x f\right)$. If we write $g(t, x) = g_\infty(x) + \phi(t, x)$, we find:

$$\begin{cases} \partial_t\phi(t,x) + a_\infty\partial_x\phi(t,x) + f(x)\phi(t,x) = -a(\phi)g'_\infty(x) - a(\phi)\partial_x\phi(t,x),\ x,t > 0 \\ \phi(t,0) = 0. \end{cases} \tag{3}$$

We define the following unbounded linear operators on $\mathcal{X}$:

$$\mathbf{A}_0\phi = -a_\infty\phi' - f\phi,\ D(\mathbf{A}_0) = \{\phi \in \mathcal{X},\ \phi' \in \mathcal{X},\ f\phi \in \mathcal{X},\ \phi(0) = 0\}, \tag{4}$$

$$\mathbf{B}\phi = -\,a(\phi)g'_\infty,\ D(\mathbf{B}) = L^1(f(x)dx), \tag{5}$$

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{B},\ D(\mathbf{A}) = D(\mathbf{A}_0). \tag{6}$$

which allows us to write (3) as $\dot{\phi} = \mathbf{A}\phi$.

3.1 Semigroup of solutions

We solve the linear equation (3) based on fairly standard tools from C_0–semigroup theory. What is noticeable in the following proposition is that the linearized equation generates a **positive** C_0–semigroup. We know [8, 18] that the nonlinear semigroup of solutions of (1) is positive. Intuitively, one can think of the linear semigroup, built in the following proposition, as the differential of the nonlinear one. Hence, we do not expect it to be positive.

Proposition 3.1 *Grant Assumption 1. Let us consider the semigroup $(\mathbf{T}_0(t))_{t\geq 0}$ on $\mathcal{X}$ given by the formula*

$$(\mathbf{T}_0(t)\phi)(x) = \exp\left(-\frac{1}{a_\infty}\int_{x-a_\infty t}^{x} f\right)\phi(x-a_\infty t)H(x-a_\infty t) \tag{7}$$

Then, we have the following properties:

1. *$(\mathbf{T}_0(t))_{t\geq 0}$ is a positive contraction C^0-semigroup on $\mathcal{X}$,*
2. *its infinitesimal generator is given by $(\mathbf{A}_0, D(\mathbf{A}_0))$,*
3. *the growth bound of $(\mathbf{T}_0(t))_{t\geq 0}$ is $\omega_0 = -\infty$, hence $\Sigma(\mathbf{A}_0) = \emptyset$,*
4. *$(\mathbf{A}, D(\mathbf{A}))$ generates a positive C^0-continuous semigroup $(\mathbf{T}(t))_{t\geq 0}$ on $\mathcal{X}$.*

Proof.

1. The semigroup / positivity properties are clear. By definition

$$\|\mathbf{T}_0(t)\phi\|_{\mathcal{X}} = \int_{a_\infty t}^{\infty} \exp\left(-\frac{1}{a_\infty}\int_{x-a_\infty t}^{x} f\right)|\phi(x-a_\infty t)|dx \leq \|\phi\|_{\mathcal{X}}\,.$$

This shows that $\mathbf{T}_0(t)$ is a contraction on $\mathcal{X}$. We now show the strong continuity (with $a_\infty = 1$ for simplicity), $\forall\phi\in\mathcal{X}$, $\forall t\geq 0$:

$$\begin{aligned}\|\mathbf{T}_0(t)\phi-\phi\|_{\mathcal{X}} &\leq \int_0^\infty \left|\exp\left(-\int_x^{x+t} f\right)\phi(x)-\phi(x+t)\right|dx + \int_0^t|\phi(x)|dx\\ &\leq \int_0^\infty|\phi(x)-\phi(x+t)|dx + \int_0^\infty\left(1-\exp\left(-\int_x^{x+t}f\right)\right)|\phi(x)|dx + \int_0^t|\phi(x)|dx.\end{aligned}$$

The last two integrals tend to zero when $t\to 0^+$ by Lebesgue's dominated convergence theorem. Hence, we focus on the first integral which is linked to the strong continuity for the right translation semigroup. Let us repeat the argument. For ϕ continuous with compact support, ϕ is uniformly continuous which implies that $\|\phi(\cdot+t)-\phi\|_\infty\to 0$. Let us denote by K a compact which contains the support of $\phi(\cdot+t)-\phi$ for $t\in[0,1]$. One then obtains that $\|\phi(\cdot+t)-\phi\|_{\mathcal{X}}\leq l(K)\,\|\phi(\cdot+t)-\phi\|_\infty\to 0$ as $t\to 0^+$. We finally conclude that the first integral tends to zero for $\phi\in\mathcal{X}$ by density in $\mathcal{X}$ of the continuous functions with compact support.

2. We start by showing that $\mu\mathrm{Id}-\mathbf{A}_0$ is injective for $\mu\in\mathbb{C}$. Let us consider $\psi\in\ker(\mu\mathrm{Id}-\mathbf{A}_0)$. Then for any $x_0>0$, one finds $\psi(x)=\exp\left(-\frac{1}{a_\infty}\int_{x_0}^x f+\mu\right)\psi(x_0)$. From $\psi(0)=0$, one gets that $\psi=0$ and $\mu\mathrm{Id}-\mathbf{A}_0$ is injective.

   As $\mathbf{T}_0$ is a contraction C^0-semigroup, the resolvent of its infinitesimal generator $\tilde{\mathbf{A}}_0$ satisfies $\mathbf{R}(\mu,\tilde{\mathbf{A}}_0) = \int_0^\infty e^{-\mu t}\mathbf{T}_0(t)dt$ for $\Re\mu>0$ and $\mathbf{R}(\mu,\tilde{\mathbf{A}}_0)\mathcal{X} = D(\tilde{\mathbf{A}}_0)$. From (7), we find the following expression

$$\psi(x) \overset{def}{=} \mathbf{R}(\mu,\tilde{\mathbf{A}}_0)\phi(x) = \frac{g_\infty(x)}{e^{\mu x/a_\infty}}\int_0^x \frac{e^{\mu y/a_\infty}}{g_\infty(y)}\frac{\phi(y)}{a_\infty}dy.$$

It follows that $\psi \in W^1_{loc}(\mathbb{R}_+)$ and $\psi(0) = 0$. Finally, using Fubini theorem, we find that for all $\mu \in \mathbb{C}_{\geq 0}$:

$$
\begin{aligned}
a_\infty \|f\psi\|_{\mathcal{X}} &\leq \int_{\mathbb{R}^2_+} dxdy\, \mathbf{1}(y \leq x) f(x) \frac{g_\infty(x)}{g_\infty(y)} e^{-\Re\mu(x-y)/a_\infty} |\phi(y)| \\
&= \int_0^\infty dy |\phi(y)| \left[\int_y^\infty f(x) \frac{g_\infty(x)}{g_\infty(y)} e^{-\Re\mu(x-y)/a_\infty} dx\right] \\
&\leq \int_0^\infty dy |\phi(y)| \left[\int_y^\infty f(x) \frac{g_\infty(x)}{g_\infty(y)} dx\right] \leq a_\infty \|\phi\|_{\mathcal{X}}.
\end{aligned}
$$

For the last equality, we used that $\int_y^\infty f(x)\frac{g_\infty(x)}{g_\infty(y)}dx = \left[-a_\infty \exp\left(-\frac{1}{a_\infty}\int_y^x f\right)\right]_y^\infty \leq a_\infty$. It implies that $f\psi \in \mathcal{X}$. Note that this last result can also be written:

$$
\left\| f\mathbf{R}(\mu, \tilde{\mathbf{A}}_0)\phi \right\|_{\mathcal{X}} = \left| a\left(\mathbf{R}(\mu, \tilde{\mathbf{A}}_0)\phi\right) \right| \leq \|\phi\|_{\mathcal{X}}. \tag{8}
$$

From the expression of ψ, we get by differentiating:

$$
a_\infty \psi' = -f\psi + \phi - \mu\psi \in \mathcal{X} \tag{9}
$$

which shows that $\psi' \in \mathcal{X}$. It follows that $(\mu\mathrm{Id} - \mathbf{A}_0)\psi = \phi$ and that

$$
D(\tilde{\mathbf{A}}_0) \subset \{\psi \in \mathcal{X},\ f\psi \in \mathcal{X},\ \psi' \in \mathcal{X},\ \psi(0) = 0\} \stackrel{def}{=} D(\mathbf{A}_0).
$$

Reciprocally, we consider $\psi \in D(\mathbf{A}_0)$ and $\mu \in \mathbb{C}_{>0}$, we write $\phi = a_\infty\psi' + f\psi + \mu\psi$. We find that $\phi \in \mathcal{X}$ and $\psi = \mathbf{R}(\mu, \tilde{\mathbf{A}}_0)\phi$ by injectivity of $\mu\mathrm{Id} - \mathbf{A}_0$. This concludes item 2 by showing that $\mathbf{A}_0 = \tilde{\mathbf{A}}_0$ with same domains.

3. We now compute the growth bound $\omega_0 \stackrel{def}{=} \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1$ such that $\|\mathbf{T}(t)\|_{\mathcal{L}(\mathcal{X})} \leq M_\omega e^{\omega t},\ \forall t \geq 0\} = \lim_{t\to\infty} \frac{1}{t}\log\|\mathbf{T}_0(t)\|_{\mathcal{L}(\mathcal{X})}$. From

$$
\|\mathbf{T}_0(t)\phi\|_{\mathcal{X}} = \int_0^\infty \exp\left(-\frac{1}{a_\infty}\int_x^{x+a_\infty t} f\right)|\phi(x)|dx,
$$

we find

$$
\|\mathbf{T}_0(t)\|_{\mathcal{L}(\mathcal{X})} = \sup_{x\geq 0}\left[\exp\left(-\frac{1}{a_\infty}\int_x^{x+a_\infty t} f\right)\right] = \exp\left(-\frac{1}{a_\infty}\int_0^{a_\infty t} f\right).
$$

Using Assumption 1, it gives $\omega_0 = \lim_{t\to\infty}\frac{1}{t}\log\|\mathbf{T}_0(t)\|_{\mathcal{L}(\mathcal{X})} = -\infty$ from which it follows that $\Sigma(\mathbf{A}_0) = \emptyset$.

4. We first note that $D(\mathbf{A}_0) \subset D(\mathbf{B})$. We compute for all $\mu \in \mathbb{C}_{>0}$

$$
\|\mathbf{B}\mathbf{R}(\mu, \mathbf{A}_0)\phi\|_{\mathcal{X}} = \|g'_\infty\|_{\mathcal{X}} \cdot |a(\mathbf{R}(\mu, \mathbf{A}_0)\phi)| \stackrel{(8)}{\leq} \|g'_\infty\|_{\mathcal{X}}\|\phi\|_{\mathcal{X}} \tag{10}
$$

which shows that

$$
\|\mathbf{B}\phi\|_{\mathcal{X}} \leq \|g'_\infty\|_{\mathcal{X}}\|(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}},\ \forall\phi \in D(\mathbf{A}_0).
$$

Similarly, one have that $\|\mathbf{A}_0\mathbf{B}\phi\|_{\mathcal{X}} = O(\|(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}})$ for $\phi \in D(\mathbf{A}_0)$. Hence, $\mathbf{B}$ is continuous on the Sobolev space $\mathcal{X}_1^{\mathbf{A}_0} \stackrel{def}{=} (D(\mathbf{A}_0), \|(\mu - \mathbf{A}_0)\cdot\|_{\mathcal{X}})$, *i.e.* $\mathbf{B} \in \mathcal{L}(\mathcal{X}_1^{\mathbf{A}_0})$. The bounded (positive) perturbation of a (positive) generator being a (positive) generator (see [17][VI.1.11]), $\mathbf{A}$ generates a positive C^0-semigroup on $\mathcal{X}_1^{\mathbf{A}_0}$. By extrapolation, this is also true on $\mathcal{X}_0^{\mathbf{A}_0} \stackrel{def}{=} \mathcal{X}$.

3.2 Spectral properties

We shall now investigate the asymptotic behavior of the solution of (3) through the analysis of the spectrum $\Sigma(\mathbf{A})$ of the infinitesimal generator $\mathbf{A}$. This is achieved in the following proposition by looking at the spectral bound $s(\mathbf{A})$ and by taking advantage of the positivity of the semigroup $(\mathbf{T}(t))_{t\geq 0}$.

Proposition 3.2 *Grant Assumption 1. The following spectral properties for the generator $\mathbf{A}$ hold true:*

1. *the spectrum of $(\mathbf{A}, D(\mathbf{A}))$ is composed of isolated eigenvalues μ solutions of*

$$\Delta(\mu) \stackrel{def}{=} 1 + a\left(\mathbf{R}(\mu, \mathbf{A}_0)g'_\infty\right) = 1 - \frac{1}{a_\infty^2}\int_0^\infty dx f(x)g_\infty(x)\int_0^x f(y)e^{-\frac{\mu}{a_\infty}(x-y)}dy = 0,$$

2. *0 is a simple eigenvalue of $\mathbf{A}$ and the spectral bound $s(\mathbf{A}) = 0$ belongs to $\Sigma(\mathbf{A})$, hence $\Sigma(\mathbf{A}) \subset \mathbb{C}_{\leq 0}$,*
3. *$\Sigma(\mathbf{A}) \cap i\mathbb{R} = \{0\}$.*

Proof.

1. Let us consider $\mu \in \mathbb{C}$. Since $\Sigma(\mathbf{A}_0) = \emptyset$, solving $(\mu \cdot \mathrm{Id} - \mathbf{A})\phi = \psi$ with $\psi \in \mathcal{X}$ is equivalent to solving $\phi - \mathbf{R}(\mu, \mathbf{A}_0)\mathbf{B}\phi = \mathbf{R}(\mu, \mathbf{A}_0)\psi$. It follows that ϕ exists if and only if $1 + a\left(\mathbf{R}(\mu, \mathbf{A}_0)g'_\infty\right) \neq 0$ which gives $\Sigma(\mathbf{A}) = \{\mu \in \mathbb{C}, 1 + a\left(\mathbf{R}(\mu, \mathbf{A}_0)g'_\infty\right) = 0\}$. The function Δ is holomorphic which implies that its zeros are isolated. Finally, the spectrum is composed of eigenvalues μ_k as one can check that the eigenvectors are given by $\mathbf{R}(\mu_k, \mathbf{A}_0)g'_\infty$ using the eigenvector equation $\phi = \mathbf{R}(\mu, \mathbf{A}_0)\mathbf{B}\phi$ for each zero μ_k of Δ. When $\mu \notin \Sigma(\mathbf{A})$, the resolvent reads:

$$\phi = \mathbf{R}(\mu, \mathbf{A})\psi = \mathbf{R}(\mu, \mathbf{A}_0)\left(\psi - \frac{a\left(\mathbf{R}(\mu, \mathbf{A}_0)\psi\right)}{1 + a\left(\mathbf{R}(\mu, \mathbf{A}_0)g'_\infty\right)}g'_\infty\right). \tag{11}$$

2. The semigroup $(\mathbf{T}(t))_{t\geq 0}$ being positive, the spectral bound $s(\mathbf{A})$ of its generator $\mathbf{A}$ belongs to the spectrum of $\mathbf{A}$: $s(\mathbf{A}) \in \Sigma(\mathbf{A}) \cap \mathbb{R}$. Hence using the previous item (1), we are looking for $s(\mathbf{A})$ as the maximal real eigenvalue. One finds that Δ is strictly increasing on $\mathbb{R}$ and that $\Delta(0) = 0$. Indeed:

$$\Delta(0) = 1 - \frac{1}{a_\infty}\int g'_\infty(y)\int_0^y f = 1 - \frac{1}{a_\infty}\int g_\infty f \stackrel{def}{=} 0.$$

Hence, $s(\mathbf{A}) = 0$. Finally $\Delta'(0) \neq 0$ implies that 0 is a simple eigenvalue.

3. From (11), the spectrum is composed of poles of the resolvent. It follows from Theorem VI-1.12 in [17] and the positivity of $(\mathbf{T}(t))_{t\geq 0}$, that the boundary spectrum $\Sigma(\mathbf{A}) \cap (s(\mathbf{A}) + i\mathbb{R})$ is cyclic, meaning that if there is $\alpha \in \mathbb{R}$ such that $s(\mathbf{A}) + i\alpha \in \Sigma(\mathbf{A})$, then $s(\mathbf{A}) + ik\alpha \in \Sigma(\mathbf{A})$ for all $k \in \mathbb{Z}$. We consider $\Delta(it)$ for $t \in \mathbb{R}$. Using Riemann-Lebesgue theorem and Lebesgue dominated theorem, we have $\Delta(it) \stackrel{t\to\pm\infty}{\longrightarrow} 1$. This implies that $\alpha = 0$ and $\Sigma(\mathbf{A}) \cap i\mathbb{R} = \{0\}$.

A numerical example of the spectrum is shown in Figure 1 Left. The fact that $0 \in \Sigma(\mathbf{A})$ is easily seen from the existence of the family of equilibria (2). The flow associated with (1), stemming from the distribution of a stochastic process, conserves the integral of g. In fact, it can be shown that this property also holds true for the semigroup $\mathbf{T}(t)$. Hence, it is convenient to define

$$\hat{\mathcal{X}} = \left\{\phi \in \mathcal{X} \mid \int_0^\infty \phi = 0\right\}.$$

Next, we compute the spectral projector associated with the zero eigenvalue. This will be useful in the last section on nonlinear stability. We recall (see Theorem III.6.17 in [27]) some basic facts about the Riesz-Dunford spectral projector. If there exists a rectifiable, simple, closed curve γ which encloses an open set containing the eigenvalue 0 in its interior and $\Sigma(\mathbf{A}) \setminus \{0\}$ in its exterior, then the Riesz-Dunford spectral projector $\mathbf{P}_0 : \mathcal{X} \to \ker(\mathbf{A})$ is defined by $\mathbf{P}_0 = \frac{1}{2i\pi}\int_\gamma \mathbf{R}(\lambda, \mathbf{A})d\lambda$. It is the unique spectral projector on $\ker(\mathbf{A})$ which commutes with $\mathbf{A}$. In our case, such γ exists because $0 \in \Sigma(\mathbf{A})$ is isolated.

Proposition 3.3 *The Riesz-Dunford spectral projector for the zero eigenvalue is*

$$\forall \phi \in \mathcal{X},\ \mathbf{P}_0\phi = \frac{I(\phi)}{I(\mathbf{R}(0, \mathbf{A}_0)g'_\infty)}\mathbf{R}(0, \mathbf{A}_0)g'_\infty.$$

Hence $Ran(Id - \mathbf{P}_0) = \widehat{\mathcal{X}}$. *Also,* $\mathbf{P}_0$ *and* $\mathbf{T}(t)$ *commute.*

Proof. Using an integration by parts, one have the following formula from Lemma B.1

$$\forall \mu \in \mathbb{C},\ \forall \phi \in \mathcal{X}, \quad a(\mathbf{R}(\mu, \mathbf{A}_0)\phi) = -\mu I(\mathbf{R}(\mu, \mathbf{A}_0)\phi) + I(\phi). \tag{12}$$

Combining the resolvent expression (11) with (12), we find $\forall \psi \in \mathcal{X}$:

$$\lim_{\lambda \to 0} \lambda \mathbf{R}(\lambda, \mathbf{A})\phi = \frac{I(\phi)}{I(\mathbf{R}(0, \mathbf{A}_0)g'_\infty)}\mathbf{R}(0, \mathbf{A}_0)g'_\infty.$$

The Riesz-Dunford projector $\mathbf{P}_0$ is the residue of $\mathbf{R}(\lambda, \mathbf{A})$ at $\lambda = 0$ which provides the expression of the projector using the above limit. The statement about the range of $\mathrm{Id} - \mathbf{P}_0$ is direct. As $\mathbf{P}_0$ can be expressed as an integral of the resolvent in the complex domain, the commutation of $\mathbf{P}_0$ and $\mathbf{T}(t)$ is a consequence of the Post–Widder Inversion Formula. Let us show it directly. $\mathbf{P}_0$ and $\mathbf{T}(t)$ commute if and only if $I(\mathbf{T}(t)\phi) = I(\phi)$ for all $t \geq 0$ and $\phi \in \mathcal{X}$. If $\phi \in D(\mathbf{A})$, one have, using an integration by parts, that $\frac{d}{dt}I(\mathbf{T}(t)\phi)) = I(\mathbf{A}\mathbf{T}(t)\phi) = 0$. Hence, $I(\mathbf{T}(t)\phi) = I(\phi)$ for $\phi \in D(\mathbf{A})$. It is then also true for $\phi \in \mathcal{X}$ by density of $D(\mathbf{A})$ in $\mathcal{X}$.

We are now ready to give the main result of this section concerning the asymptotic behavior of the linear equation (3).

Theorem 3.1 *There is a spectral decomposition of* $\mathcal{X}$ *into flow invariant subspaces:*

$$\mathcal{X} = \mathbb{R} \cdot \mathbf{e} \oplus \widehat{\mathcal{X}}$$

associated with the projector $\mathbf{P}_0$ *where* $\mathbf{e} \overset{def}{=} \mathbf{R}(0, \mathbf{A}_0)g'_\infty$ *is an eigenvector for the eigenvalue* 0*. We write* $\mathbf{A}_|$ *(resp.* $(\mathbf{T}_|(t))_{t\geq 0}$*) the part of* $\mathbf{A}$ *(resp.* $(\mathbf{T}(t))_{t\geq 0}$*) in* $\widehat{\mathcal{X}}$*. One has* $s(\mathbf{A}_|) < 0$ *and* $(\mathbf{T}_|(t))_{t\geq 0}$ *is uniformly exponentially stable i.e. for every positive* ϵ *small enough, there is a constant* $M_\epsilon \geq 1$ *such that for all* $t \geq 0$

$$\|\mathbf{T}(t) - \mathbf{P}_0\|_{\mathcal{L}(\mathcal{X})} = \left\|\mathbf{T}_|(t)\right\|_{\mathcal{L}(\mathcal{X})} \leq M_\epsilon e^{(s(\mathbf{A}_|)+\epsilon)t}. \tag{13}$$

Finally, we have the following result concerning the spectral radius $\sup\limits_{\lambda \in \Sigma(\mathbf{T}_|(t))} |\lambda| = e^{s(\mathbf{A}_|)t} < 1.$

Proof. The spectral decomposition into spaces invariant by $\mathbf{A}$ is a consequence of the previous proposition concerning the Riesz-Dunford projector and of [27] Theorem 6.17. This theorem also implies that $\Sigma(\mathbf{A}_|) = \Sigma(\mathbf{A}) \setminus \{0\}$ whence $\sup \Re\Sigma(\mathbf{A}_|) < 0$. Also, $\mathbf{P}_0$ commutes with $\mathbf{T}(t)$ so that the semigroup $\mathbf{T}_|(t)$ belongs to $\mathcal{L}(\widehat{\mathcal{X}})$. Hence, the subspaces are flow invariant. We now prove that the spectral bound $s(\mathbf{A}_|)$ equals the growth bound $\omega_0(\mathbf{T}_|)$. This is a consequence of Theorem-12.17 in [2, 10] as $\hat{\mathcal{X}}$ is an AL-space, *i.e.* the norm satisfies $\|\phi_1 + \phi_2\| = \|\phi_1\| + \|\phi_2\|$ for all $\phi_1, \phi_2 \in \hat{\mathcal{X}}_+$, and $(\mathbf{T}_|(t))$ is a positive semigroup on $\hat{\mathcal{X}}$. This gives the formula (13).

As $\omega_0(\mathbf{T}_|) = s(\mathbf{A}_|)$, for all $\epsilon > 0$ small enough, there is a constant $M_\epsilon \geq 1$ such that $\left\|\mathbf{T}_|(t)\right\| \leq M_\epsilon e^{(s(\mathbf{A}_|)+\epsilon)t}$. The Gelfand spectral radius theorem [16][VII.3.4] gives $\sup\limits_{\lambda \in \Sigma(\mathbf{T}_|(t))} |\lambda| = \lim\limits_n \sqrt[n]{\left\|\mathbf{T}_|(nt)\right\|} \leq e^{(s(\mathbf{A}_|)+\epsilon)t}$. As ϵ is arbitrary, this gives $\sup\limits_{\lambda \in \Sigma(\mathbf{T}_|(t))} |\lambda| \leq e^{s(\mathbf{A}_|)t}$. The equality follows from the existence of an eigenvalue λ_1 such that $\Re\lambda_1 = s(\mathbf{A}_|)$.

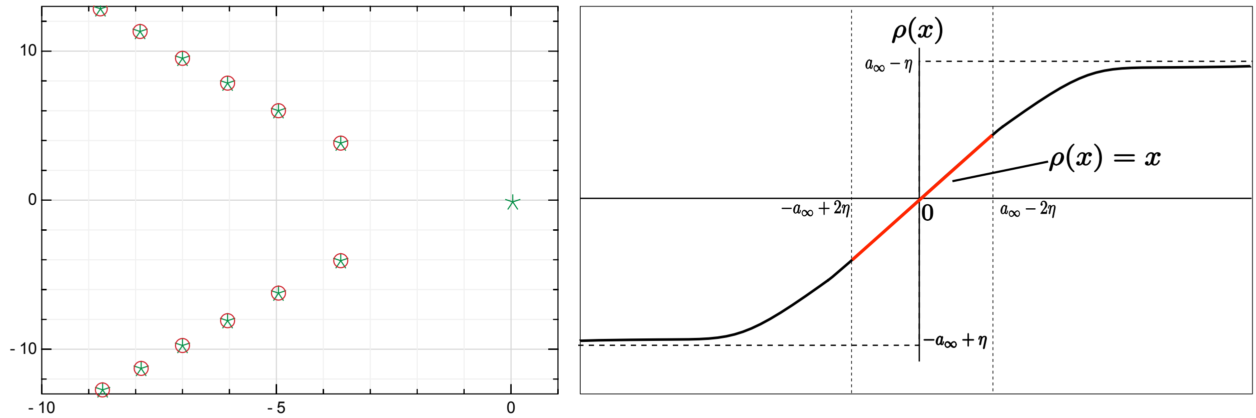

Figure 1: Left: Rightmost part of the spectrum of $\mathbf{A}$ (green) and $\mathbf{A}_|$ (red) for $f(x) = x^2$. Computed using collocation methods provided by the *Julia* package *ApproxFun.jl* (see [29]). Right: Plot of the cut-off function ρ.

3.3 Sobolev spaces

We collect here some results concerning the Sobolev spaces associated with $\mathbf{A}$. This is very helpful as climbing up the Sobolev spaces of $\mathbf{A}$, solutions gain regularity while the asymptotic properties of the semigroup remain the same. However, the Sobolev norm for $\mathbf{A}_0$ is much simpler than the one for $\mathbf{A}$ and this is why we spend some time relating the Sobolev spaces of $\mathbf{A}$ and $\mathbf{A}_0$.

Lemma 3.1 *Grant Assumption 1. For the operators $\mathbf{A}_0$ and $\mathbf{A}$ defined in (4), (6), we have the following properties:*

1. *for $n \in \{1,2\}$, $\mathcal{X}_n^{\mathbf{A}} = \mathcal{X}_n^{\mathbf{A}_0}$ with equivalent norms,*
2. *for $n \in \{1,2\}$, $\mathbf{A}$ restricted to $\mathcal{X}_n^{\mathbf{A}_0}$ generates a C^0–semigroup,*
3. *we have: $\mathcal{X}_1^{\mathbf{A}_0} = \{\phi \in \mathcal{X},\ \phi' \in \mathcal{X},\ f\phi \in \mathcal{X},\ \phi(0) = 0\}$ endowed with the norm $\|\cdot\|_{1,\mathbf{A}_0} = \|\mathbf{A}_0 \cdot\|_{\mathcal{X}}$. The $\mathcal{X}_1^{\mathbf{A}_0}$–norm is equivalent to the norm*
$$\|\phi\|_1 = \|\phi\|_{\mathcal{X}} + \|\phi'\|_{\mathcal{X}} + \|f\phi\|_{\mathcal{X}},$$
4. *we have: $\mathcal{X}_2^{\mathbf{A}_0} = \{\phi \in \mathcal{X},\ \phi'' \in \mathcal{X},\ f\phi' \in \mathcal{X},\ f^2\phi \in \mathcal{X},\ \phi(0) = \phi'(0) = 0\}$ endowed with the norm $\|\cdot\|_{2,\mathbf{A}_0} = \|\mathbf{A}_0^2 \cdot\|_{\mathcal{X}}$. The $\mathcal{X}_2^{\mathbf{A}_0}$–norm is equivalent to the norm*
$$\|\phi\|_2 = \|\phi\|_{\mathcal{X}} + \|f^2\phi\|_{\mathcal{X}} + \|f\phi'\|_{\mathcal{X}} + \|\phi''\|_{\mathcal{X}},$$
5. *for $n \in \{1,2\}$, the Sobolev spaces $\left(\mathcal{X}_n^{\mathbf{C}_\alpha}\right)_{\alpha>0}$ for $\mathbf{C}_\alpha = -\partial_x - \alpha\mathbf{M}_f$ are the same, with equivalent norms.*

Proof. See appendix D.
To shorten notations, since the $\mathcal{X}_n^{\mathbf{A}_0}$ and $\mathcal{X}_n^{\mathbf{A}}$ norms are equivalent for $n \in \{1,2\}$, we write $\|\cdot\|_{n,\mathbf{A}} = \|\mathbf{A}_0^n \cdot\|_{\mathcal{X}}$.

4 Nonlinear stability

This section establishes the local exponential convergence of the solution g of (1) to g_∞, for all initial conditions $g_0 = g_\infty + \phi$ with ϕ close to zero and of zero integral. The proof also works for any g_α. This result improves on some points those in [18] where it was shown that $\|g(t) - g_\infty\|_{L^1} = O\left((1+t)^{-1/\xi}\right)$ if $f(x) \geq x^\xi$ for all $x \in [0,1]$ where $c > 0, \xi \geq 1$, and for all initial condition $g_0 \in L^1_+(\mathbb{R}_+)$ of integral one such that $g(0) = 1,\ g_0 \in C^1_b(\mathbb{R}_+)$, $\int_0^\infty f^2 g_0 < \infty$ and $\int_0^\infty |g_0'| < \infty$.

4.1 Difficulties and strategy

The general strategy is similar to that in [11, 25]: we apply a Picard like iteration scheme to the nonlinear semigroup of solutions at some time t_0 to show that it converges to a fixed point. To this end, we need to build a nonlinear semigroup of solutions of (1) which is differentiable. Such semigroup can be found in [18] but the differentiability was not investigated. Here we construct a semigroup using a different method by means of a fixed point argument based on the computation of the instantaneous rate function $a(g(t))$. Doing so requires g to be integrable against f. The Picard iteration additionally requires g to be integrable against f^2 similar to the requirement mentioned above at the beginning of the section. The smallest Sobolev space satisfying this is $\mathcal{X}_2^{\mathbf{A}_0}$ in which we solve (1). The second requirement in applying [11, 25] is differentiability of the nonlinear semigroup. However, the nonlinear flow of (1) is not differentiable in $\mathcal{X}_2^{\mathbf{A}_0}$. Indeed, from [18] or using the method of characteristics, its (implicit) expression can be found to be:

$$g(t,x) = \exp\left(\int_{\beta_t(x)}^t -f(\varphi_{\beta_t(x),s}(0))ds\right) 1_{x\leq A(t)} \\ + g_0(x-A(t)) \exp\left(\int_0^t -f(\varphi_{0,s}(x-A(t)))ds\right) 1_{x>A(t)}$$

with $A(t) = \int_0^t a$, $\varphi_{s,t}(x) = x + \int_s^t a$ and $\beta_t(x)$ such that $\int_{\beta_t(x)}^t a = x$ for $x \leq A(t)$. Moreover $a(t)$ solves the fixed point equation $a(t) = \int_0^\infty fg(t)$. One can show that for $T > 0$, the mapping $\varphi \to a$ is C^1 from a neighborhood of g_∞ in $\mathcal{X}_2^{\mathbf{A}}$ into $C^0([0,T])$. However, for all $t > 0$, the mapping $\varphi \to \varphi(\cdot - A(t))$ is not even Lipschitz from $\mathcal{X}_2^{\mathbf{A}}$ to itself and so is the flow as well. To overcome this problem and inspired by [20], we perform a change of variable in time in (1). Roughly speaking, we set $h(\tau(t),x) = g(t,x)$ with $\tau = \int_0^t a(g)(s)ds$. This change of variable is possible only if $\tau(t)$ is invertible or equivalently if $t \to a(g)(t)$ is strictly positive. Hence, we modify the vector field in order to insure that this condition is met. We then show that this defines a new flow which is differentiable and which enables to characterize the asymptotic behavior of the initial one.

4.2 Time rescaling

In order to perform a time rescaling, we introduce the following cut-off function which is strictly positive and locally identical to $a(g)$ if this latter is close enough to a_∞:

$$\widetilde{a}(g) = a_\infty + \rho_\eta(a(g) - a_\infty), \quad \text{with} \quad \begin{cases} \rho_\eta(x) = x \quad \text{if} \quad |x| \leq a_\infty - 2\eta \\ |\rho_\eta(x)| \leq a_\infty - \eta \quad \forall x \in \mathbb{R} \\ \rho_\eta \in C^1(\mathbb{R}) \quad \text{non decreasing and} \quad \|\rho'_\eta\|_{C^0(\mathbb{R})} < \infty \end{cases} \tag{14}$$

where η is a constant such that $0 < \eta < \frac{a_\infty}{2}$ (see Figure 1). We shall write ρ for ρ_η when no confusion is possible. Note that whenever possible, we also write $\widetilde{a}(t)$ for $\widetilde{a}(g(t))$ or for $a_\infty + \rho_\eta(a(t))$ (in case $a \in C^0(\mathbb{R}_+,\mathbb{R})$). We have

$$0 < \underline{a} \overset{def}{=} \eta \leq \widetilde{a}(t) \leq 2a_\infty - \eta \overset{def}{=} \bar{a}. \tag{15}$$

Let us now formally perform the time rescaling:

$$h(\tau(t),x) \overset{def}{=} \widetilde{g}(t,x) \quad \text{with} \quad \tau(t) = \int_0^t \widetilde{a}(\widetilde{g}(s,\cdot))ds,$$

where $\widetilde{g}$ is solution of (1) upon replacing $a(g)$ by $\widetilde{a}(g)$. Thanks to the cutoff, τ is invertible and $h(\tau,x)$ solves

$$\begin{cases} \partial_\tau h(\tau,x) + \partial_x h(\tau,x) = -\frac{f(x)}{\widetilde{a}(h(\tau,\cdot))} h(\tau,x), \quad x,\tau > 0 \\ h(t,0) = 1. \end{cases} \tag{16}$$

We remove the boundary condition by translating the problem around g_∞, $h = g_\infty + u$, it gives:

$$\begin{cases} \partial_\tau u(\tau,x) = -\partial_x u(\tau,x) - \frac{f(x)u(\tau,x)}{a_\infty+\rho(a(u(\tau)))} + \left(\frac{1}{a_\infty} - \frac{1}{a_\infty+\rho(a(u(\tau)))}\right) f g_\infty, \quad x,\tau > 0 \\ u(\tau,0) = 0. \end{cases} \tag{17}$$

After this formal time rescaling, we plan to prove the differentiability of the nonlinear semigroup associated with the flow of (17). In section 4.3, we set the mathematical framework for the analysis of (17) and prove the existence of the nonlinear semigroup as follows. First, we consider the non-autonomous problem on $\mathcal{X}$:

$$\begin{cases} \dot{u}(t) = \mathbf{A}(t)u(t),\ t > s \geq 0 \\ u(s) = \varphi \end{cases} \tag{NAH}$$

$$\mathbf{A}(t)\phi = -\phi' - \frac{f\phi}{a_\infty + \rho(a(t))},\ D(\mathbf{A}(t)) = \mathcal{X}_1^{\mathbf{A}_0} \tag{18}$$

for $a \in C^0([s,T])$. We show the well-posedness of (NAH) in the sense that it admits a $\mathcal{X}_2^{\mathbf{A}}$*–valued solution* (Definition 3) written $u(t) = \mathbf{U}_a(t,s)\varphi$. Then, we consider the inhomogeneous problem:

$$\begin{cases} \dot{u}(t) = \mathbf{A}(t)u(t) + g_a(t),\ t > s \geq 0 \\ u(s) = \varphi \end{cases} \tag{NAIH}$$

with

$$g_a(t) \stackrel{def}{=} \left(\frac{1}{a_\infty} - \frac{1}{a_\infty + \rho(a(t))}\right) f g_\infty. \tag{19}$$

We show that it admits a $\mathcal{X}_2^{\mathbf{A}}$–valued solution $u(t) = \mathbf{V}_a(t,s)\varphi$. In a last step, we establish the existence and uniqueness of a solution of the fixed point equation $a(t) = \int_0^\infty f\mathbf{V}_a(t,s)\varphi$ in $C_b^0([s,\infty))$ for $\varphi \in \mathcal{X}_2^{\mathbf{A}}$ and conclude by the existence of the nonlinear semigroup namely $(\mathbf{S}_r(t))_{t\geq 0}$ associated with the flow of (NAIH) with $a(t)$ solution of this fixed point equation. In section 4.4, we show the Fréchet differentiability of $\mathbf{S}_r(t)$ and establish the local exponential stability of 0. Finally, in section 4.5, we link the asymptotic behavior of the solution of the rescaled problem (17) to the initial one (1).

4.3 Solution of the rescaled equation

For $s \geq 0$ and $a \in C^0([s,\infty))$, we introduce a family of bounded operators $(\mathbf{U}_a(t,s))_{t\geq s}$ on $\mathcal{X}$ defined by

$$(\mathbf{U}_a(t,s)\varphi)(x) \stackrel{def}{=} \exp\left(-\int_s^t \frac{f(v+x-t)}{a_\infty + \rho(a(v))}dv\right) H(x-t+s)\varphi(x-t+s), \quad \forall\varphi \in \mathcal{X}. \tag{20}$$

For $\bar{a} > 0$, up to some abuse of notation, we also define the following contraction C_0-semigroup $(\mathbf{U}_{\bar{a}}(t))_{t\geq 0}$ on $\mathcal{X}$:

$$(\mathbf{U}_{\bar{a}}(t)\phi)(x) \stackrel{def}{=} \exp\left(-\frac{1}{\bar{a}}\int_{x-t}^x f\right) H(x-t)\phi(x-t)$$

with generator* $(\bar{\mathbf{A}}, \mathcal{X}_1^{\mathbf{A}})$ where $\bar{\mathbf{A}}\phi = -\phi' - \frac{1}{\bar{a}}f\phi$ (see Proposition 3.2). Finally, we introduce the solution of the inhomogeneous problem (NAIH)

$$\mathbf{V}_a(t,s)\varphi \stackrel{def}{=} \mathbf{U}_a(t,s)\varphi + \int_s^t \mathbf{U}_a(t,r)g_a(r)dr,\ t \geq s. \tag{21}$$

*Actually its domain is $\mathcal{X}_1^{\bar{\mathbf{A}}}$ but $\mathcal{X}_1^{\bar{\mathbf{A}}} = \mathcal{X}_1^{\mathbf{A}}$ by Lemma 3.1.

Remark 2
- *From (15), we find $\mathbf{U}_a(t,s) \le \mathbf{U}_{\bar{a}}(t-s)$.*
- *Let us note that a is seen through the cutoff in the semigroups $\mathbf{U}_a(t,s)$, $\mathbf{V}_a(t,s)$ and the function g_a. Hence $\mathbf{U}_a(t,s)$ and $\mathbf{V}_a(t,s)$ are well defined for $t \ge s \ge 0$ and $a \in C^0(\mathbb{R}^+, \mathbb{R})$.*

The following proposition establishes the well-posedness of (NAH) as there is an evolution family which solves (NAH) in the space $\mathcal{X}_2^{\mathbf{A}}$ *i.e.* it leaves $\mathcal{X}_2^{\mathbf{A}}$ invariant. Moreover this solution also belongs to the smaller space $C^0(\mathbb{R}_+, \mathcal{X}_2^{\mathbf{A}})$ in effect giving an $\mathcal{X}_2^{\mathbf{A}}$-valued solution (see Definition 3).

Proposition 4.1 *Grant Assumption 1. For $s \ge 0$, let $a \in C^0([s,\infty))$, then $(\mathbf{U}_a(t,s))_{t\ge s}$ is an **evolution family** of contractions on $\mathcal{X}$ which solves the Cauchy problem (NAH) on $\mathcal{X}_2^{\mathbf{A}}$. Moreover for $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, $\mathbf{U}_a(t,s)\varphi$ is a $\mathcal{X}_2^{\mathbf{A}}$–valued solution of the initial value problem (NAH) and there is a constant $C > 0$, independent of a, such that $\forall (t,s) \in \mathbb{R}_+^2,\ t \ge s,\ \forall \phi \in \mathcal{X}_2^{\mathbf{A}}$:*

$$\|\mathbf{U}_a(t,s)\varphi\|_{2,\mathbf{A}} \le C\|\varphi\|_{2,\mathbf{A}}. \tag{22}$$

Proof. The proof of the fact that $(\mathbf{U}_a(t,s))_{t\ge s}$ is an evolution family of contractions on $\mathcal{X}$ is direct. We focus on showing that $\mathbf{U}_a(t,s)\varphi$ is a $\mathcal{X}_2^{\mathbf{A}}$–valued solution for $\varphi \in \mathcal{X}_2^{\mathbf{A}}$ which implies that it solves (NAH) on $\mathcal{X}_2^{\mathbf{A}}$. We first note that $\mathcal{X}_2^{\mathbf{A}}$ is densely and continuously embedded in $\mathcal{X}$ and that $\mathcal{X}_2^{\mathbf{A}} \subset D(\mathbf{A}(t))$ as a consequence of Lemma 3.1 and of the fact that $\widetilde{a}$ is positive bounded with values in $[\underline{a}, \bar{a}]$.

First step. Let us prove that for $0 \le s \le t$, $\mathbf{U}_a(t,s)\mathcal{X}_2^{\mathbf{A}} \subset \mathcal{X}_2^{\mathbf{A}}$. It is indeed needed to identify a subset of the domain of $\mathbf{A}(t)$ to define a (classical) solution. This is a consequence of Lemma E.2 from which it also follows that there is a constant $C > 0$ such that for all $t \ge s \ge 0$ and $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, $\|\mathbf{U}_a(t,s)\varphi\|_{2,\mathbf{A}} \le C\|\varphi\|_{2,\mathbf{A}}$. Hence, $\mathbf{U}_a(t,s)$ leaves $\mathcal{X}_2^{\mathbf{A}}$ invariant and is bounded on $\mathcal{X}_2^{\mathbf{A}}$.

Second step. We show the strong continuity of the family on $\mathcal{X}_2^{\mathbf{A}}$ which is useful in the fourth step of the proof. One needs to show that $\forall \varphi \in \mathcal{X}_2^{\mathbf{A}}$, $\|\mathbf{U}_a(t',s')\varphi - \mathbf{U}_a(t,s)\varphi\|_{2,\mathbf{A}} \to 0$ when $(t',s') \to (t,s)$. By dominating the terms $\mathbf{U}_a(t,s)\varphi$, $f^2\mathbf{U}_a(t,s)\varphi$, $f(\mathbf{U}_a(t,s)\varphi)'$ and $(\mathbf{U}_a(t,s)\varphi)''$ as done in the proof of Lemma E.2, and using Lebesgue dominated convergence, we obtain the strong continuity. In particular, this yields an evolution family on $\mathcal{X}_2^{\mathbf{A}}$ and $t \to \mathbf{U}_a(t,s)\varphi \in C^0([s,\infty), \mathcal{X}_2^{\mathbf{A}})$.

Third step. For $t > s \ge 0$ and $\phi \in \mathcal{X}_2^{\mathbf{A}}$, we write $u(t,x) \overset{def}{=} (\mathbf{U}_a(t,s))(x)$ which we decompose as $u(t,x) = q(t,s,x)v(t,x)$ where $v(t,x) \overset{def}{=} H(x-t+s)\phi(x-t+s) = (\mathbf{T}_r(t-s)\phi)(x)$ stems from the right translation semigroup. We note that $v(t)$ is the classical solution of $\dot{v} = -\partial_x v$ in $\mathcal{X}$ such that $v(t,0) = 0$, hence v belongs to $C^1((s,\infty), \mathcal{X})$. As q is C^1 in x, t, we find $\partial_t q(t,s,x) = -\frac{f(x)}{\widetilde{a}(t)} q(t,s,x) - \partial_x q(t,s,x)$. It gives $\dot{u} = -\frac{fu}{\widetilde{a}(t)} - v\partial_x q - q\partial_x v$ showing that $u = qv$ solves (NAH) in $\mathcal{X}$. The fact that v belongs to $C^1((s,\infty), \mathcal{X})$ implies the same for u. Finally, $u \in C^0([s,\infty), \mathcal{X}_2^{\mathbf{A}})$ is a consequence of the strong continuity on $\mathcal{X}_2^{\mathbf{A}}$. This shows that u is a $\mathcal{X}_2^{\mathbf{A}}$-valued solution of (NAH). Finally, (22) was proved in Lemma E.2.

Proposition 4.2 *Grant Assumptions 1 and 2. Let $s \ge 0$, $a \in C^0([s,\infty))$ and $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, then (NAIH) has a unique $\mathcal{X}_2^{\mathbf{A}}$-valued solution given by (21). Moreover there exists a constant $C > 0$ independent of a such that for all $t \ge s \ge 0$ and $\varphi \in \mathcal{X}_2^{\mathbf{A}}$:*

$$\|\mathbf{V}_a(t,s)\varphi\|_{2,\mathbf{A}} \le C\left(\|a\|_{C^0([s,t])} + \|\varphi\|_{2,\mathbf{A}}\right). \tag{23}$$

Proof. This is an adaptation of the proof of Theorem V.5.2 in [32]. We first note that $g_a \in C^0(\mathbb{R}_+, \mathcal{X}_2^{\mathbf{A}})$ under Assumption 2. An $\mathcal{X}_2^{\mathbf{A}}$-solution starting at time s from $\phi \in \mathcal{X}_2^{\mathbf{A}} \subset D(\mathbf{A}(s))$ is a classical solution. In particular, it is an integral solution whose expression is given by $\mathbf{V}_a(t,s)$ (see [32]).

Reciprocally, we only need to check that $u(t) = \int_s^t \mathbf{U}_a(t,r)g_a(r)dr$ is a $\mathcal{X}_2^{\mathbf{A}}$-valued solution since $\mathbf{U}_a(t,s)\varphi$ is a $\mathcal{X}_2^{\mathbf{A}}$-valued solution of the homogeneous problem. From the strong continuity of $\mathbf{U}_a$ on $\mathcal{X}_2^{\mathbf{A}}$ as shown in the proof of Proposition 4.1, we find that $r \to \mathbf{U}_a(t,r)g_a(r)$ is continuous in $\mathcal{X}_2^{\mathbf{A}}$ which implies that $t \to u(t)$ is continuous in $\mathcal{X}_2^{\mathbf{A}}$. Then, from $\mathbf{A}(t) - \mathbf{A}(s) = -\left(\frac{1}{\widetilde{a}(t)} - \frac{1}{\widetilde{a}(s)}\right)\mathbf{M}_f$, we conclude that $\mathbf{A} \in C^0(\mathbb{R}_+, \mathcal{L}(\mathcal{X}_2^{\mathbf{A}}, \mathcal{X}))$ using Lemma C.1.

It follows that $r \to \mathbf{A}(t)\mathbf{U}_a(t,r)g_a(r)$ is continuous in $\mathcal{X}$ which implies that $t \to u(t)$ is continuously differentiable in $\mathcal{X}$ and that[†] $\frac{d}{dt}u(t) = \mathbf{A}(t)u(t) + g_a(t)$ holds in $\mathcal{X}$ for $t \in [s,\infty)$.

From (35b), there is a constant $C > 0$ independent of a such that

$$\|\mathbf{V}_a(t,s)\varphi\|_{2,\mathbf{A}} \leq C(\|\rho(a)\|_{C^0([s,t])} + \|\varphi\|_{2,\mathbf{A}}) \leq C(\|a\|_{C^0([s,t])}\,\|\rho'\|_\infty + \|\varphi\|_{2,\mathbf{A}}) \lesssim \left(\|a\|_{C^0([s,t])} + \|\varphi\|_{2,\mathbf{A}}\right)$$

which yields the inequality (23).

The sequel of this section is devoted to solving the Volterra-like fixed point equation $a(t) = \int_0^\infty f\mathbf{V}_a(t,s)\varphi$ in some Banach space that we shall now precise. For $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, we introduce the mapping $\mathcal{T}_{s,\varphi}$:

$$\mathcal{T}_{s,\varphi}\ :\ \begin{array}{ccc} C^0([s,\infty)) & \longrightarrow & C_b^0([s,\infty)) \\ c & \longrightarrow & a(\mathbf{V}_c(\cdot,s)\varphi). \end{array} \tag{24}$$

Proposition 4.3 *Grant Assumptions 1 and 2. There exists $C > 0$ such that for all $\varphi \in \mathcal{X}_2^{\mathbf{A}}$ and for all $s \geq 0$, the mapping $\mathcal{T}_{s,\varphi}$ is a contraction on $C^0([s,s+\delta])$ provided that $0 < \delta < 1$ and that $\delta < C\left(1 + \|\varphi\|_{2,\mathbf{A}}\right)^{-1}$.*

Proof. For $s \geq 0, \delta > 0$ and $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, let us show that $\mathcal{T}_{s,\varphi}$ leaves $C^0([s,s+\delta])$ invariant. For $a \in C^0([s,s+\delta])$, Proposition 4.2 implies that $t \to \mathbf{V}_a(t,s)\varphi$ belongs to $C^0([s,s+\delta],\mathcal{X}_2^{\mathbf{A}})$. By continuity of $\mathbf{M}_f$ from $\mathcal{X}_2^{\mathbf{A}}$ to $\mathcal{X}$, we find that $\mathcal{T}_{s,\varphi}(a) \in C^0([s,s+\delta])$. For $\delta > 0$, we estimate the C^0–norm of $\mathcal{T}_{s,\varphi}(a_2) - \mathcal{T}_{s,\varphi}(a_1)$ using (21) for $a_1, a_2 \in C^0([s,s+\delta])$:

$$\mathcal{T}_{s,\varphi}(a_2)(t) - \mathcal{T}_{s,\varphi}(a_1)(t) = a((\mathbf{U}_{a_2}(t,s) - \mathbf{U}_{a_1}(t,s))\varphi) + a\Big(\int_s^t \mathbf{U}_{a_2}(t,r)(g_{a_2}(r) - g_{a_1}(r))dr\Big) + a\Big(\int_s^t (\mathbf{U}_{a_2}(t,r) - \mathbf{U}_{a_1}(t,r))g_{a_1}(r)dr\Big).$$

Using that $0 < \underline{a} \leq \widetilde{a} \leq \bar{a}$ from the definition of the cut off (14), we find

$$|(\mathbf{U}_{a_2}(t,s) - \mathbf{U}_{a_1}(t,s))\varphi| \leq \frac{1}{\underline{a}^2}(t-s)\|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])} f\mathbf{U}_{\bar{a}}(t-s)|\varphi|.$$

From the above inequality and Remark 1, we get two bounds:

$$\begin{aligned} a((\mathbf{U}_{a_2}(t,s) - \mathbf{U}_{a_1}(t,s))\varphi) &\leq \frac{1}{\underline{a}^2}(t-s)\|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])}\|f^2\mathbf{U}_{\bar{a}}(t-s)|\varphi|\|_{\mathcal{X}} \\ &\lesssim (1 + \|\varphi\|_{\mathcal{X}} + \|f^2\varphi\|_{\mathcal{X}})(t-s)\|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])} \\ a\Big(\int_s^t (\mathbf{U}_{a_2}(t,r) - \mathbf{U}_{a_1}(t,r))g_{a_1}(r)dr\Big) &\leq \frac{1}{\underline{a}^2}\|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])} a\Big(f\int_s^t (t-r)\mathbf{U}_{\bar{a}}(t-r)g_{a_1}(r)dr\Big), \\ &\lesssim \|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])}(t-s)^2\left\|f^3 g_\infty\right\|_{\mathcal{X}} \lesssim \|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])}(t-s)^2 \\ &\overset{\delta<1}{\lesssim} \|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])}(t-s). \end{aligned}$$

Similarly

$$\begin{aligned} a\Big(\int_s^t \mathbf{U}_{a_2}(t,r)(g_{a_2}(r) - g_{a_1}(r))dr\Big) &\lesssim \|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])} a\Big(\int_s^t \mathbf{U}_{\bar{a}}(t-r)(fg_\infty)dr\Big) \\ &\lesssim \|\widetilde{a}_2 - \widetilde{a}_1\|_{C^0([s,s+\delta])}(t-s)\left\|f^2 g_\infty\right\|_{\mathcal{X}}. \end{aligned}$$

[†]One can use the identity for $h \geq 0$, $(\mathbf{U}_a(t+h,t) - Id)u(t) = u(t+h) - u(t) - \int_t^{t+h} \mathbf{U}_a(t+h,r)g_a(r)dr$

Hence, if $\delta < 1$, the Lipschitz constant $k(\varphi)$ of $\mathcal{T}_{s,\varphi}$ on $C^0([s, s+\delta])$ reads $k(\varphi) = C\left(1 + \|\varphi\|_{\mathcal{X}} + \|f^2\varphi\|_{\mathcal{X}}\right)\delta \overset{\text{Lemma C.1}}{\leq} C(1 + \|\varphi\|_{2,\mathbf{A}})\delta$ with C independent of φ, s and δ. It goes to zero when $\delta \to 0$. We can thus cw hoose δ for $\mathcal{T}_{s,\varphi}$ to be a contraction.

Theorem 4.1 *Grant Assumptions 1 and 2. For each $\varphi \in \mathcal{X}_2^{\mathbf{A}}$ and $s \geq 0$, there is a unique solution $a \in C_b^0([s,\infty))$ of $\mathcal{T}_{s,\varphi}(a) = a$. Moreover, $\phi : t \to \mathbf{V}_a(t,s)\varphi$ belongs to $C_b^0([s,\infty), \mathcal{X}_2^{\mathbf{A}}) \cap C^1((s,\infty), \mathcal{X})$ and solves*

$$\begin{cases} \partial_t\phi + \partial_x\phi = -\dfrac{f\phi}{\widetilde{a}(g_\infty + \phi)} + f g_\infty \left(\dfrac{1}{\widetilde{a}(g_\infty + \phi)} - \dfrac{1}{a_\infty}\right), \; t > s, \; x > 0, \\ \qquad \phi(s) = \varphi. \end{cases} \tag{25}$$

Proof. Let $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, Proposition 4.3 and the Picard Theorem give the existence of an increasing sequence $(s_n)_{n\in\mathbb{N}}$ such that the differences $s_i - s_{i-1}$ satisfy

$$s_i - s_{i-1} = \min\left(\frac{C}{2}(1 + \|\varphi_{i-1}\|_{2,\mathbf{A}})^{-1}, \frac{1}{2}\right), \; s_0 = s$$

where C is the constant from Proposition 4.3 and $\varphi_0 \overset{def}{=} \varphi$, $\varphi_{i+1} \overset{def}{=} \mathbf{V}_{a_{i+1}}(s_{i+1}, s_i)\varphi_i$ with a_{i+1} solution of $a_{i+1} = \mathcal{T}_{s_i,\varphi_i}(a_{i+1})$ in $C^0([s_i, s_{i+1}])$ for $i \geq 0$. For $i \geq 1$, we note that

$$\begin{aligned} a_{i+1}(s_i) &= \mathcal{T}_{s_i,\varphi_i}(a_{i+1})(s_i) = a(\varphi_i) \\ &= a(\mathbf{V}_{a_i}(s_i, s_{i-1})\varphi_{i-1}) = \mathcal{T}_{s_{i-1},\varphi_{i-1}}(a_i)(s_i) = a_i(s_i). \end{aligned}$$

Hence, if we define $a \overset{def}{=} a_i$ on $[s_{i-1}, s_i), \quad \forall i \geq 1$, we have that $a \in C^0([s, \lim_n s_n))$. For $i \geq 1$ and $t \in [s_{i-1}, s_i]$, we have

$$\begin{aligned} \mathcal{T}_{s,\varphi}(a)(t) &= a(\mathbf{V}_a(t,s)\varphi) = a(\mathbf{V}_a(t, s_{i-1})\varphi_{i-1}) = \mathcal{T}_{s_{i-1},\varphi_{i-1}}(a_i)(t) \\ &= a_i(t) = a(t). \end{aligned}$$

Hence, a is the unique fixed point of $\mathcal{T}_{s,\varphi}$ in $C^0([s, \lim_n s_n))$. It follows that $\varphi_i = \mathbf{V}_a(s_i, s)\varphi_0$ and $\forall i \geq 1, \; \|\varphi_i\|_{2,\mathbf{A}} = \|\mathbf{V}_a(s_i, s)\varphi_0\|_{2,\mathbf{A}} \overset{(35b)}{\leq} C_V(1 + \|\varphi_0\|_{2,\mathbf{A}})$. Hence, $s_i - s_{i-1} \geq \min\left(\frac{1}{2}, C(1 + \|\varphi_0\|_{2,\mathbf{A}})^{-1}\right)$ for some new constant $C > 0$ and $\lim_n s_n = +\infty$.

The boundedness of a results from (35b): $a \in C_b^0([s,\infty))$. Let us define $\phi(t) \overset{def}{=} \mathbf{V}_a(t,s)\varphi$ which solves (thanks to Proposition 4.2) the problem (NAIH) with the denominator $a_\infty + \rho(a(t)) = a_\infty + \rho(a(\phi(t))) = \widetilde{a}(g_\infty + \phi(t))$ by definition of the fixed point a. Then ϕ solves the problem (25) as a $\mathcal{X}_2^{\mathbf{A}}$–valued solution (*i.e.* it belongs to $C_b^0([s,\infty), \mathcal{X}_2^{\mathbf{A}}) \cap C^1(]s,\infty), \mathcal{X})$).

Based on the previous theorem, we introduce the mapping:

$$\begin{array}{cccc} & \mathcal{X}_2^{\mathbf{A}} & \to & C_b^0([0,\infty)) \\ \mathcal{A}: & \varphi & \to & \mathcal{A}(\varphi) \end{array} \tag{26}$$

where $\mathcal{A}(\phi)$ is the fixed point of $\mathcal{T}_{0,\varphi}$. Note that $\mathcal{A}(0) = 0$. We also define the nonlinear semigroup $(\mathbf{S}_r(t))_{t\geq 0}$ for the **rescaled** equation (25):

$$\begin{array}{cccc} & \mathcal{X}_2^{\mathbf{A}} & \to & \mathcal{X}_2^{\mathbf{A}} \\ \mathbf{S}_r(t): & \varphi & \to & \mathbf{V}_{\mathcal{A}(\varphi)}(t,0)\varphi. \end{array} \tag{27}$$

The function $t \to \mathbf{S}_r(t)\varphi$ is the unique $\mathcal{X}_2^{\mathbf{A}}$-valued solution of (25) with $s = 0$. Note that it satisfies $\mathbf{S}_r(0) = 0$.

4.4 Convergence to the equilibrium g_∞

We now study the differentiability of the nonlinear semigroup $\mathbf{S}_r$. We start with the regularity of the co-restriction of the map $\mathcal{A}$ with values in $C^0([0,t])$ for $t>0$.

Lemma 4.1 *Grant Assumptions 1 and 2. For $t>0$ small enough, there is a neighborhood $\mathcal{V}\subset\mathcal{X}_2^{\mathbf{A}}$ of 0 such that $\mathcal{A}$ belongs to $C^1(\mathcal{V},C^0([0,s]))$ for all $s\in[0,t]$. Moreover, for all $\phi\in\mathbf{P}_0\mathcal{V}$ and for all $s\in[0,t]$, $\mathcal{A}(\phi)$ is such that $I\left(\mathbf{V}_{\mathcal{A}(\varphi)}(s,0)\varphi\right)=0$ meaning that $\mathbf{S}_r(s)\phi\in\widehat{\mathcal{X}}_2^{\mathbf{A}}$.*

Proof. We note that $\mathcal{T}:(a,\phi)\to\mathcal{T}_{0,\phi}(a)$ belongs to $C^1(C^0([0,t])\times\mathcal{X}_2^{\mathbf{A}},C^0([0,t]))$ as consequence of Proposition E.1 and of the continuity of $\mathbf{M}_f$ from $\mathcal{X}_2^{\mathbf{A}}$ to $\mathcal{X}$. We wish to apply the parametrized contracting mapping theorem (see [22]). For $t<1$ small enough, one can chose $R>0$ such that $a\to\mathcal{T}(a,\phi)$ is a C^1 family of contractions with Lipschitz constant $k(\phi)$ in the first variable for $\phi\in B_{\mathcal{X}_2^{\mathbf{A}}}(0,R)$. We have:

$$k(\phi)\overset{Prop.\ 4.3}{\leq}tC(1+R)<1.$$

As a consequence of Theorem 3.2 in [22], $\mathcal{A}(\phi)$, fixed point of $\mathcal{T}(\cdot,\phi)$, is C^1 from $B_{\mathcal{X}_2^{\mathbf{A}}}(0,R)$ into $C^0([0,t])$ and the same holds on $B_{\widehat{\mathcal{X}}_2^{\mathbf{A}}}(0,R)$. This concludes the first part of the proof (the slightly more general result in the lemma is straightforward).

We wish to show how a neighborhood of 0 in $\mathcal{X}_2^{\mathbf{A}}$ is mapped into a neighborhood of 0 in $C^0([0,t])$. We use the fact that $\mathcal{A}$ is a Lipschitz as we now show. For all $\phi,\psi\in B_{\mathcal{X}_2^{\mathbf{A}}}(0,R)$:

$$\begin{aligned}\|\mathcal{A}(\phi)-\mathcal{A}(\psi)\|&\leq\|\mathcal{T}_{0,\phi}(\mathcal{A}(\phi))-T_{0,\phi}(\mathcal{A}(\psi))\|+\|\mathcal{T}_{0,\phi}(\mathcal{A}(\psi))-T_{0,\psi}(\mathcal{A}(\psi))\|\\&\leq tC(1+R)\left\|\mathcal{A}(\phi)-\mathcal{A}(\psi)\right\|+\|\mathcal{T}_{0,\phi}(\mathcal{A}(\psi))-T_{0,\psi}(\mathcal{A}(\psi))\|\\&\overset{Lemma\ D}{\leq}tC(1+R)\left\|\mathcal{A}(\phi)-\mathcal{A}(\psi)\right\|+C\left\|\phi-\psi\right\|_{2,\mathbf{A}}\end{aligned}$$

which gives $\|\mathcal{A}(\phi)-\mathcal{A}(\psi)\|\leq\frac{C}{1-tC(1+R)}\left\|\phi-\psi\right\|_{2,\mathbf{A}}$ and $\mathcal{A}$ is Lipschitz on $B_{\mathcal{X}_2^{\mathbf{A}}}(0,R)$.

Let then $\mathcal{V}\subset B_{\mathcal{X}_2^{\mathbf{A}}}(0,R)$ be small enough ensuring that $\mathcal{A}$ maps $\mathcal{V}$ into $B_{C^0}(0,r)$ with $r>0$ such that the cut-off ρ (14) satisfies $\rho(x)=x$ for $|x|\leq r$. It implies that $\mathcal{A}(\phi)=\rho(\mathcal{A}(\phi))$. For $\phi\in\mathbf{P}_0\mathcal{V}$, we write $I(t)=I(\mathbf{V}_{\mathcal{A}(\varphi)}(t,0)\varphi)$ and $v(t)\overset{def}{=}\mathbf{V}_{\mathcal{A}(\varphi)}(t,0)\varphi$. By hypothesis: $I(0)=0$. As $v\in C^1((0,\infty),\mathcal{X})$, we have that $\frac{d}{dt}I(v(t))=I(\dot v(t))$ for all $t>0$ and from Theorem 4.1:

$$\begin{aligned}\frac{d}{dt}I(t)&=I(\dot v(t))=I(\mathbf{A}(t)v(t)+g_{\mathcal{A}(\varphi)}(t))\\&=I\left(-v'(t)-\frac{fv(t)}{a_\infty+\mathcal{A}(\varphi)(t)}-\left(\frac{1}{a_\infty+\mathcal{A}(\varphi)(t)}-\frac{1}{a_\infty}\right)fg_\infty\right)\\&\overset{I.B.P.}{=}-[v(\infty,t)-v(0,t)]-\frac{I(fv(t))}{a_\infty+\mathcal{A}(\varphi)(t)}+\frac{\mathcal{A}(\varphi)(t)}{a_\infty+\mathcal{A}(\varphi)(t)}=\frac{\mathcal{A}(\varphi)(t)-\mathcal{A}(\varphi)(t)}{a_\infty+\mathcal{A}(\varphi)(t)}=0\end{aligned}$$

where we wrote $v(\infty,t)=\lim_{x\to\infty}v(x,t)$ which is zero because $v(\cdot,t)\in W^{1,1}(\mathbb{R}_+)$. It follows that $I(t)$ is constant for $t>0$ hence equal to zero by continuity. This concludes the proof of the second part.

Proposition 4.4 *Grant Assumptions 1 and 2. For $t>0$ small enough, let $\mathbf{P}_0\mathcal{V}$ be the neighborhood of 0 in $\widehat{\mathcal{X}}_2^{\mathbf{A}}$ introduced in Lemma 4.1. The nonlinear semigroup $\mathbf{S}_r(s)$ evaluated at time $s\in[0,t]$, belongs to $C^1(\mathbf{P}_0\mathcal{V},\widehat{\mathcal{X}}_2^{\mathbf{A}})$ and the Fréchet-differential of $\mathbf{S}_r(s)|_{\widehat{\mathcal{X}}_2^{\mathbf{A}}}$ at 0 is $\mathbf{T}_|(s/a_\infty)$. Finally, there is a constant $C\geq1$ such that $\forall s\in[0,t]$, $\forall\varphi\in\mathcal{V}$:*

$$\|\mathbf{S}_r(s)\varphi\|_{2,\mathbf{A}}\leq C\|\varphi\|_{2,\mathbf{A}}.\tag{28}$$

Proof. We first show that $\mathbf{S}_r(s)$ is differentiable. By Lemma 4.1, there exists $t > 0$ small enough and $\mathcal{V} \subset \mathcal{X}_2^{\mathbf{A}}$ such that $\mathcal{A} \in C^1(\mathcal{V}, C^0([0,t]))$ and $I(\mathbf{V}_{\mathcal{A}(\varphi)}(s,0)\varphi) = 0$ for all $\varphi \in \mathbf{P}_0\mathcal{V}$ and $s \in [0,t]$. Moreover for $\varphi \in \mathcal{X}_2^{\mathbf{A}}$ and $s \in [0,t]$, the mapping $a \to \mathbf{V}_a(s,0)\varphi$ belongs to $C^1(C^0([0,t]), \mathcal{X}_2^{\mathbf{A}})$ by Proposition E.1 and the mapping $\varphi \to \mathbf{V}_a(s,0)\varphi$ is affine so is differentiable. By composition we deduce that $\forall s \in [0,t]$, $\mathbf{S}_r(s) \in C^1(\mathcal{V}, \mathcal{X}_2^{\mathbf{A}})$. Moreover, for $s \in [0,t]$ $\mathbf{S}_r(s)\mathbf{P}_0\mathcal{V} \subset \widehat{\mathcal{X}}_2^{\mathbf{A}}$ which gives $d[\mathbf{S}_r(s)](0) \in \mathcal{L}(\widehat{\mathcal{X}}_2^{\mathbf{A}})$.

Let us now show that the Fréchet differential of $\mathbf{S}_r(s)|_{\widehat{\mathcal{X}}_2^{\mathbf{A}}}$ at point 0 is $\mathbf{T}_|(s/a_\infty)$. We first note that $\mathbf{U}_{\mathcal{A}(\phi)}(t,0)\phi = \mathbf{U}_{a_\infty}(t,0)\phi + o(\phi)$. By differentiating $\mathcal{A}(\phi) = a(\mathbf{S}_r(\cdot)\phi)$, we obtain $d\mathcal{A}(0)\varphi = a(d[\mathbf{S}_r(\cdot)](0)\varphi) = a(u(\cdot))$ where $u(s) \overset{def}{=} d[\mathbf{S}_r(s)](0)\varphi$ for all $s \in [0,t]$. By differentiating $\varphi \to \mathbf{S}_r(s)\varphi = \mathbf{V}_{\mathcal{A}(\varphi)}(s,0)\varphi$ at 0, we obtain from (21) that $\forall s \in [0,t],\ \phi \in \mathbf{P}_0\mathcal{V}$,

$$\begin{aligned} u(s) &\overset{def}{=} d[\mathbf{S}_r(s)](0)\varphi = \mathbf{U}_{a_\infty}(s,0)\varphi + \int_0^s \mathbf{U}_{a_\infty}(s,r)\left(\frac{fg_\infty}{a_\infty^2}\right) d\mathcal{A}(0)(\phi)(r)dr \\ &= \mathbf{T}_0\left(\frac{s}{a_\infty}\right)\varphi - \int_0^s \mathbf{T}_0\left(\frac{s-r}{a_\infty}\right)\left(\frac{g'_\infty}{a_\infty}\right) a(u(r))dr = \mathbf{T}_0\left(\frac{s}{a_\infty}\right)\varphi + \frac{1}{a_\infty}\int_0^s \mathbf{T}_0\left(\frac{s-r}{a_\infty}\right)\mathbf{B}(u(r))dr. \end{aligned}$$

We conclude that $u(s) = \mathbf{T}(s/a_\infty)\varphi$ from the uniqueness of the solution of (3). The fact that $\mathbf{S}_r(s)|_{\widehat{\mathcal{X}}_2^{\mathbf{A}}}$ is defined on $\widehat{\mathcal{X}}_2^{\mathbf{A}}$ implies that its differential is the part $\mathbf{T}_|(s/a_\infty)$ of $\mathbf{T}(s/a_\infty)$ in $\widehat{\mathcal{X}}_2^{\mathbf{A}}$. Noting that $\mathcal{A}(0) = 0$, the inequality (28) is obtained from (23) and using the fact that $\mathcal{A}$ is Lipschitz as shown in the proof of the previous proposition.

We are now ready to study the long term behavior of $\mathbf{S}_r$.

Theorem 4.2 *Grant Assumptions 1 and 2. The stationary solution* 0 *of* (NAIH) *is locally exponentially stable in* $\widehat{\mathcal{X}}_2^{\mathbf{A}}$ *that is for all* $\epsilon > 0$ *small enough, there is a neighborhood* $\mathcal{V}_\epsilon \subset \widehat{\mathcal{X}}_2^{\mathbf{A}}$ *such that*

$$\exists C_\epsilon \geq 1 \quad \forall \phi \in \mathcal{V}_\epsilon,\ \forall t \geq 0\ \ \|\mathbf{S}_r(t)\phi\|_{\mathcal{X}_2^{\mathbf{A}}} \leq C_\epsilon e^{\left(\frac{s(\mathbf{A}_|)}{a_\infty}+\epsilon\right)t}\|\phi\|_{2,\mathbf{A}}. \tag{29}$$

Proof. Using the notations of Proposition 4.4, let $\mathcal{V} \subset \mathcal{X}_2^{\mathbf{A}}$ such that $\mathbf{S}_r(s) \in C^1(\mathbf{P}_0\mathcal{V}; \widehat{\mathcal{X}}_2^{\mathbf{A}})$. The Fréchet differential of $\mathbf{S}_r(s)|_{\widehat{\mathcal{X}}_2^{\mathbf{A}}}$ at 0 is $\mathbf{T}_|(s/a_\infty) \in \mathcal{L}(\widehat{\mathcal{X}}_2^{\mathbf{A}})$ and its spectrum lies in a compact subset of the open unit disc, see Theorem 3.1. The theorem also provides the spectral radius of $\mathbf{T}_|(s/a_\infty)$. We deduce from Theorem A.1 that for $\epsilon > 0$ small enough, there is a neighborhood $\mathcal{V}' = B(0,R)$ of 0 in $\widehat{\mathcal{X}}_2^{\mathbf{A}}$ and a constant $C \geq 1$ such that

$$\forall \phi \in \mathcal{V}',\ \forall n \in \mathbb{N} \quad \|\mathbf{S}_r(s)^n\phi\|_{2,\mathbf{A}} = \|\mathbf{S}_r(ns)\phi\|_{2,\mathbf{A}} \leq C\left(\kappa+\epsilon\right)^n \|\phi\|_{2,\mathbf{A}}$$

where $\kappa \overset{def}{=} e^{\frac{s(\mathbf{A}_|)}{a_\infty}s} \in (0,1)$. Moreover, we have that $\mathbf{S}_r(s) : \mathcal{V}' \to \mathcal{V}'$ by Theorem A.1. We define $\mathcal{V}_\epsilon \overset{def}{=} \{\phi \in \mathcal{V}',\ \|\phi\|_{2,\mathbf{A}} \leq R/C_L\}$ where C_L is the constant in (28). It follows that $\forall \phi \in \mathcal{V}_\epsilon$ and $\forall q \in [0,s)$, $\mathbf{S}_r(q)\phi \in \mathcal{V}'$. We can thus decompose each $t \geq 0$ as $t = ns + q$ with $q \in [0,s)$ and find $\mathbf{S}_r(t) = \mathbf{S}_r(ns)\mathbf{S}_r(q)$. It follows that:

$$\forall \phi \in \mathcal{V}_\epsilon,\ \|\mathbf{S}_r(t)\phi\|_{2,\mathbf{A}} \leq C(\kappa+\epsilon)^n\|\mathbf{S}_r(q)\phi\|_{2,\mathbf{A}} \overset{(28)}{\leq} C'(\kappa+\epsilon)^n\|\phi\|_{2,\mathbf{A}}.$$

Finally, we note that $(\kappa+\epsilon)^n \leq \kappa^n e^{n\epsilon/\kappa}$ and up to renaming ϵ, there is a constant C, independent of n, t, such that $\|\mathbf{S}_r(t)\phi\|_{2,\mathbf{A}} \leq Ce^{\left(\frac{s(\mathbf{A}_|)}{a_\infty}+\epsilon\right)t}\|\phi\|_{2,\mathbf{A}}$.

4.5 Main result

In this section, we conclude with the main result concerning the nonlinear stability of (1).

Theorem 4.3 *Grant Assumptions 1 and 2. The equilibrium* 0 *is locally exponentially stable with respect to* $(\mathbf{S}(t))_{t\geq 0}$ *in* $\widehat{\mathcal{X}}_2^{\mathbf{A}}$.

Proof. Using Lemma 4.1 and the fact that $\mathcal{A}$ is Lipschitz, there is a neighborhood $\mathcal{W} \subset \mathbf{P}_0\mathcal{V}$ of 0 in $\widehat{\mathcal{X}}_2^{\mathbf{A}}$ satisfying $\mathcal{A}(\mathcal{W}) \subset B_{C^0}(0, r)$ with r such that $\rho(x) = x$ for $|x| \leq r$. Using Theorem 4.2, $\forall\epsilon > 0$, there is an open ball $B(0, R_\epsilon) \subset \mathcal{W}$ such that $\forall t \geq 0$, $\mathbf{S}_r(t)B(0, R_\epsilon) \subset \mathcal{W}$. Hence, $(\mathbf{S}_r(t))_{t\geq 0}$ solves (25) on $B(0, R_\epsilon)$ but with $\widetilde{a}$ replaced by a.

As a consequence, for $\phi \in B(0, R_\epsilon)$, the function $t_\phi(\tau) = \int_0^\tau \frac{1}{a(\mathbf{S}_r(s)\phi)}ds$ for $\tau \geq 0$ is well-defined, positive, monotone and invertible. We can thus define $\mathbf{S}(\tau)\phi \overset{def}{=} g_\infty + \mathbf{S}_r(t_\phi^{-1}(\tau))\phi$ for all $\tau \geq 0$. It follows that $(\mathbf{S}(t))_{t\geq 0}$ solves (1).

Thanks to Theorem 4.2, we have $\forall \phi \in B(0, R_\epsilon)$, $\forall t \geq 0$ $\|\mathbf{S}_r(t)\phi\|_{2,\mathbf{A}} \leq C_\epsilon e^{\left(\frac{s(\mathbf{A}_|)}{a_\infty}+\epsilon\right)t}\|\phi\|_{2,\mathbf{A}}$. Then, we have:

$$\|\mathbf{S}(t)\phi - g_\infty\|_{2,\mathbf{A}} \leq C_\epsilon e^{\left(\frac{s(\mathbf{A}_|)}{a_\infty}+\epsilon\right)t_\phi^{-1}(t)} \leq C_\epsilon e^{\left(\frac{s(\mathbf{A}_|)}{a_\infty}+\epsilon\right)\eta t}$$

where we used that $\eta t \leq t_\phi^{-1}(t) \leq \bar{a}t$ for $\eta > 0$, see (14). This concludes the proof.

The parameter $0 < \eta < a_\infty$ entering in the definition of the cutoff ρ_η is arbitrary. Hence, we find that the exponential convergence of $\mathbf{S}(t)$ is $C_\epsilon e^{\left(s(\mathbf{A}_|)+\epsilon\right)t}$ with $\epsilon > 0$ small enough.

5 Discussion

In this work, we looked at the exponential stability of a recent mean-field limit using tools from dynamical systems. This was made possible thanks to the surprising but helpful positivity of the linearized semigroup and using a time rescaling trick from [20]. This allowed us to avoid using the center manifold theory which comes up naturally for this kind of equations because of the family of equilibria.

Note that our framework does not work in the general case $\lambda > 0$. What's more, recent numerical evidence [14] suggests the existence of a Hopf bifurcation and probably of a center manifold. Thus, the present work hints at the difficulties for fulfilling such program.

Nevertheless, the present formalism allows us to look at more general situations when for example the spatial location of the neurons or propagation delays are taken into account [15, 37].

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A Definitions and results on general Cauchy problems

To be self-content, this section presents results taken from [32] that we use to show the well-posedness of (NAH) and (NAIH). We start this section with some definitions about the linear non-autonomous initial value Cauchy problem

$$\begin{cases} \dot{u}(t) = \mathbf{A}(t)u(t) \quad \text{for } 0 \leq s < t, \\ u(s) = v \end{cases} \tag{nACP}$$

on a Banach space $\mathcal{X}$.

Definition 1 *[32] An $\mathcal{X}$-valued function $u : [s,T] \to \mathcal{X}$ is called a* ***classical solution*** *of (nACP) if u is continuous on $[s,T]$, $u(t) \in D(\mathbf{A}(t))$ for $0 < s \le T$, u is continuously differentiable for $0 < s \le T$ and it satisfies (nACP).*

To discuss basic properties of (nACP), we introduce the so-called evolution semigroup associated with it.

Definition 2 *[32] A family of* ***bounded*** *operators $(\mathbf{U}(t,s))_{t \ge s}$ on a Banach space $\mathcal{X}$ is called a strongly continuous evolution family if*

(i) *$\mathbf{U}(t,s) = \mathbf{U}(t,r)\mathbf{U}(r,s)$ and $\mathbf{U}(s,s) = Id$ for $t \ge r \ge s \ge 0$ and*

(ii) *the mapping $\{(\tau,\sigma) \in \mathbb{R}^2 : \tau \ge \sigma \ge 0\} \ni (t,s) \to \mathbf{U}(t,s)$ is strongly continuous meaning that $\forall \phi \in \mathcal{X}$, $\|\mathbf{U}(t',s')\phi - \mathbf{U}(t,s)\phi\| \to 0$ as $(t',s') \to (t,s)$.*

Definition 3 *Let $\mathcal{Y} \subset \mathcal{X}$ a Banach space that is densely and continuously embedded in $\mathcal{X}$. A function $u \in C^0([s,T],\mathcal{Y})$ is a* ***$\mathcal{Y}$-valued function*** *of the initial valued problem (nACP) if $u \in C^1(]s,T],\mathcal{X})$ and (nACP) is satisfied in $\mathcal{X}$.*

The following theorem states a sufficient condition for exponential stability of a stationary solution of a discrete dynamical system.

Theorem A.1 *Let $\mathcal{X}$ be a Banach and $\mathcal{V}$ be a neighborhood of 0 in $\mathcal{X}$. Let $\mathbf{F} : \mathcal{V} \to \mathcal{X}$ be differentiable at 0 and satisfy $\mathbf{F}(0) = 0$. Let $d\mathbf{F}(0) = \mathbf{L} \in \mathcal{L}(\mathcal{X})$ be its Fréchet derivative at 0. Assume that the spectrum of $\mathbf{L}$ lies in a compact subset of the open unit disc. Then for all $\epsilon > 0$ small enough, there is a neighborhood $\mathcal{U}_\epsilon \subset \mathcal{V}$ of 0 and a constant $C_\epsilon \ge 1$ such that for all x in $\mathcal{U}_\epsilon$ and $n \in \mathbb{N}$:*

$$\|\mathbf{F}^n(x)\| \le C_\epsilon \,(b+\epsilon)^n \,\|x\|$$

where $b \overset{def}{=} \sup_{\lambda \in \Sigma(\mathbf{L})} |\lambda| < 1$. Moreover, $\mathcal{U}_\epsilon$ is invariant by $\mathbf{F}$.

Proof. (Adaptation of Theorem I.1 in [25]) Let $\epsilon > 0$ be small enough such that $b + \epsilon < 1$. There is an equivalent norm $\|\cdot\|_h$ satisfying $\|\cdot\| \le \|\cdot\|_h \le \alpha \|\cdot\|$ and such that $\|\mathbf{L}x\|_h \le (b + \epsilon/2)\,\|x\|_h$ (see [25]). The differentiability of $\mathbf{F}$ implies that there is a neighborhood $\mathcal{U}_h = \{x \in \mathcal{X}\ ;\ \|x\|_h \le R\}$ of 0 such that for $x \in \mathcal{U}_h$:

$$\|\mathbf{F}(x)\|_h \le \|\mathbf{L}x\|_h + \epsilon/2\,\|x\|_h \le (b+\epsilon)\,\|x\|_h < \|x\|_h\,.$$

Hence, $\mathbf{F}$ leaves $\mathcal{U}_h$ invariant. It follows that $\|\mathbf{F}(x)^n\| \le C(b+\epsilon)^n\,\|x\|$ for some $C \ge 1$. We now define $\mathcal{U} = \{x \in \mathcal{X}\ ;\ \|x\| \le R/\alpha\} \subset \mathcal{U}_h$. This set is invariant by $\mathbf{F}$ because $\alpha \ge 1$. This completes the proof of the theorem.

B An equality for computing the spectral projector

Lemma B.1 *We have the following identity*

$$\forall \mu \in \mathbb{C},\ \forall \phi \in \mathcal{X}, \quad a(\mathbf{R}(\mu,\mathbf{A}_0)\phi) = -\mu I(\mathbf{R}(\mu,\mathbf{A}_0)\phi) + I(\phi).$$

Proof. We start from $a_\infty a(\mathbf{R}(\mu,\mathbf{A}_0)\phi) = \int_0^\infty dy e^{\mu y/a_\infty} \frac{\phi(y)}{g_\infty(y)} \int_y^\infty dx f(x) e^{-\mu x/a_\infty} g_\infty(x)$ and use

$$\int_y^\infty dx f(x) e^{-\mu x/a_\infty} g_\infty(x) = -a_\infty \int_y^\infty dx e^{-\mu x/a_\infty} g'_\infty(x) = a_\infty g_\infty(y) e^{-\mu y/a_\infty} - \mu \int_y^\infty dx g_\infty(x) e^{-\mu x/a_\infty}$$

which gives $a_\infty a(\mathbf{R}(\mu,\mathbf{A}_0)\phi) = a_\infty I(\phi) - a_\infty I(\mathbf{R}(\mu,\mathbf{A}_0)\phi)$ as claimed.

C Continuity of M_f

This section is dedicated to the proof of the continuity of $(\mathbf{M}_f)^n : \phi \to f^n\phi$ from $\mathcal{X}_n^{\mathbf{A}_0}$ into $\mathcal{X}$.

Lemma C.1 *Assume that hypothesis 1 is satisfied. For $n \in \{1,2\}$, the linear operator $(\mathbf{M}_f)^n : \phi \to f^n\phi$ is continuous from $\mathcal{X}_n^{\mathbf{A}_0}$ into $\mathcal{X}$.*

Proof. From (8) in the proof of Proposition 3.1, we find $\mathbf{M}_f \in \mathcal{L}(\mathcal{X}_1^{\mathbf{A}_0}, \mathcal{X})$. The case $n = 2$ is similar as we now show. Take $\psi = \mathbf{R}(0, \mathbf{A}_0)^2\phi$, and write $\mathbf{R} = \mathbf{R}(0, \mathbf{A}_0)$ for simplicity

$$a_\infty \|f^2\psi\|_{\mathcal{X}} \leq \int_{\mathbb{R}_+^2} dxdy\, \mathbf{1}(y \leq x) f^2(x) \frac{g_\infty(x)}{g_\infty(y)} |\mathbf{R}\phi(y)| = \int_0^\infty dy |\mathbf{R}\phi(y)| \int_y^\infty f^2(x) \frac{g_\infty(x)}{g_\infty(y)} dx$$
$$= \begin{cases} \int_0^1 dy|\mathbf{R}\phi(y)| \int_y^\infty f^2(x)\frac{g_\infty(x)}{g_\infty(y)}dx & \overset{(30)}{\lesssim} \|\mathbf{R}\phi\|_{\mathcal{X}} \\ + & \\ \int_1^\infty dy|\mathbf{R}\phi(y)| \int_y^\infty f^2(x)\frac{g_\infty(x)}{g_\infty(y)} & \overset{(31)}{\lesssim} (\|f\mathbf{R}\phi\|_{\mathcal{X}} + \|\mathbf{R}\phi\|_{\mathcal{X}}) \end{cases}$$

For the first inequality, we used

$$\int_0^1 dy |\mathbf{R}\phi(y)| \int_y^\infty dx f^2(x) \frac{g_\infty(x)}{g_\infty(y)} \leq \frac{1}{g_\infty(1)} \int_0^1 dy |\mathbf{R}\phi(y)| \underbrace{\int_0^\infty f^2(x) g_\infty(x) dx}_{<\infty} \lesssim \|\mathbf{R}\phi\|_{\mathcal{X}}. \tag{30}$$

For the second inequality, we used that $\forall y \geq 1$:

$$\int_y^\infty f^2(x) \frac{g_\infty(x)}{g_\infty(y)} dx = \left[-a_\infty f(x) \exp\left(-\frac{1}{a_\infty} \int_y^x f \right) \right]_y^\infty + a_\infty \int_y^\infty f'(x) \exp\left(-\frac{1}{a_\infty} \int_y^x f \right)$$
$$\overset{Assumption\ 1}{\leq} a_\infty f(y) + a_\infty c \int_y^\infty f(x) \exp\left(-\frac{1}{a_\infty} \int_y^x f \right) \lesssim f(y) + g_\infty(y) \lesssim f(y) + 1. \tag{31}$$

Hence, we find that $\left\|f^2\mathbf{R}^2\phi\right\|_{\mathcal{X}} \lesssim (\|\mathbf{R}\phi\|_{\mathcal{X}} + \|f\mathbf{R}\phi\|_{\mathcal{X}}) \lesssim \|\mathbf{R}\phi\|_{\mathcal{X}}$ where the last inequality comes from the continuity of $\mathbf{M}_f$ (case $n = 1$). It follows that $\mathbf{M}_f^2 \in \mathcal{L}(\mathcal{X}_2^{\mathbf{A}_0}, \mathcal{X})$ as $\mathcal{X}_2^{\mathbf{A}_0}$ is continuously embedded in $\mathcal{X}_1^{\mathbf{A}_0}$.

D Sobolev tower: proof of Lemma 3.1

Proof. [Proof of Lemma 3.1] We prove each item separately.

Proof of item 1 Let us show that $\mathcal{X}_n^{\mathbf{A}} = \mathcal{X}_n^{\mathbf{A}_0}$[‡] where we recall that $\mathcal{X}_n^{\mathbf{A}} \overset{def}{=} (D(\mathbf{A}^n), \|\cdot\|_{n,\mathbf{A}})$ is endowed with $\|\cdot\|_{n,\mathbf{A}} = \|(\mu - \mathbf{A})^n \cdot \|_{\mathcal{X}}$ for $\mu \in \rho(\mathbf{A})$, *i.e.* $\Re\mu > 0$.

For $n = 1$, we have $D(\mathbf{A}) = D(\mathbf{A}_0)$. As for the norms, $\forall \phi \in \mathcal{X}_1^{\mathbf{A}}$ and $\mu \in \rho(\mathbf{A})$ one finds

$$\|\phi\|_{1,\mathbf{A}} \overset{def}{=} \|(\mu - \mathbf{A})\phi\|_{\mathcal{X}} = \|(\mathrm{Id} - \mathbf{BR}(\mu, \mathbf{A}_0))(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}}$$
$$\leq \|\mathrm{Id} - \mathbf{BR}(\mu, \mathbf{A}_0)\|_{\mathcal{L}(\mathcal{X})} \|(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}} \lesssim \|\phi\|_{1,\mathbf{A}_0}$$

[‡] meaning that $D(\mathbf{A}^n) = D(\mathbf{A}_0^n)$ and $\|\cdot\|_{n,\mathbf{A}} \sim \|\cdot\|_{n,\mathbf{A}_0}$

where the fact that $\mathbf{BR}(\mu, \mathbf{A}_0) \in \mathcal{L}(\mathcal{X})$ was proved in Proposition 3.1. The other inequality reads:

$$\begin{aligned}\|\phi\|_{1,\mathbf{A}_0} &\overset{def}{=} \|(\mu - \mathbf{A}_0)\phi\|_{\mathcal{X}} = \|(\mu - \mathbf{A} + \mathbf{B})\phi\|_{\mathcal{X}} = \|(\mathrm{Id} + \mathbf{BR}(\mu, \mathbf{A}))(\mu - \mathbf{A})\phi\|_{\mathcal{X}} \\ &\leq \|\mathrm{Id} + \mathbf{BR}(\mu, \mathbf{A})\|_{\mathcal{L}(\mathcal{X})} \|(\mu - \mathbf{A})\phi\|_{\mathcal{X}} \lesssim \|\phi\|_{1,\mathbf{A}}\end{aligned}$$

where $\mathbf{BR}(\mu, \mathbf{A}) \in \mathcal{L}(\mathcal{X})$ thanks to (11) and (10).

For $n = 2$:

$$\begin{aligned}D(\mathbf{A}^2) = \{\phi \in D(\mathbf{A}),\ \mathbf{A}\phi \in D(\mathbf{A})\} &= \{\phi \in D(\mathbf{A}_0),\ \mathbf{A}\phi \in D(\mathbf{A}_0)\} = \{\phi \in D(\mathbf{A}_0),\ (\mathbf{A}_0 + \mathbf{B})\phi \in D(\mathbf{A}_0)\} \\ &\overset{\mathbf{B}\phi \in D(\mathbf{A}_0)}{=} \{\phi \in D(\mathbf{A}_0),\ \mathbf{A}_0\phi \in D(\mathbf{A}_0)\} = D(\mathbf{A}_0^2).\end{aligned}$$

Concerning the norms, $\forall \phi \in \mathcal{X}_2^{\mathbf{A}}$ and $\mu \in \rho(\mathbf{A})$:

$$\begin{aligned}\|\phi\|_{2,\mathbf{A}} \overset{def}{=} \|(\mu - \mathbf{A})\phi\|_{1,\mathbf{A}} &\overset{"n=1"}{\leq} C_1\|(\mu - \mathbf{A})\phi\|_{1,\mathbf{A}_0} = C_1\|(\mathrm{Id} - \mathbf{BR}(\mu, \mathbf{A}_0))(\mu - \mathbf{A}_0)\phi\|_{1,\mathbf{A}_0} \\ &\leq C_1\|\mathrm{Id} - \mathbf{BR}(\mu, \mathbf{A}_0)\|_{\mathcal{L}(\mathcal{X}_1^{\mathbf{A}_0})}\|(\mu - \mathbf{A}_0)\phi\|_{1,\mathbf{A}_0} \lesssim \|\phi\|_{2,\mathbf{A}_0}.\end{aligned}$$

The last inequality comes from $\mathbf{B} \in \mathcal{L}(\mathcal{X}_1^{\mathbf{A}_0})$ in Proposition 3.1. For $\phi \in \mathcal{X}_1^{\mathbf{A}}$ and $\mu \in \rho(\mathbf{A})$, we find $|a(\mathbf{R}(\mu, \mathbf{A})\phi)| \lesssim \|\phi\|_{\mathcal{X}} \lesssim \|\phi\|_{\mathcal{X}_1^{\mathbf{A}}}$ using (11) and (10). Hence $\mathbf{BR}(\mu, \mathbf{A}) \in \mathcal{L}(\mathcal{X}_1^{\mathbf{A}})$ since $g'_\infty \in D(\mathbf{A}) = D(\mathbf{A}_0)$. Using this, we find:

$$\begin{aligned}\|\phi\|_{2,\mathbf{A}_0} &\overset{def}{=} \|(\mu - \mathbf{A}_0)\phi\|_{1,\mathbf{A}_0} \overset{"n=1"}{\leq} C_1\|(\mu - \mathbf{A} + \mathbf{B})\phi\|_{1,\mathbf{A}} = C_1\|(\mathrm{Id} + \mathbf{BR}(\mu, \mathbf{A}))(\mu - \mathbf{A})\phi\|_{1,\mathbf{A}} \\ &\leq C_1\|\mathrm{Id} + \mathbf{BR}(\mu, \mathbf{A})\|_{\mathcal{L}(\mathcal{X}_1^{\mathbf{A}})}\|(\mu - \mathbf{A})\phi\|_{1,\mathbf{A}} \lesssim \|\phi\|_{2,\mathbf{A}}.\end{aligned}$$

We conclude that $\mathcal{X}_n^{\mathbf{A}} = \mathcal{X}_n^{\mathbf{A}_0}$ for $n \in \{1, 2\}$ with equivalent norms.

In this proof, we endowed $\mathcal{X}_n^{\mathbf{A}_0}$ with the norm $\|(\mu - \mathbf{A}_0)^n \cdot\|$ in order to show equivalence between the $\mathcal{X}_n^{\mathbf{A}}$-norm and the $\mathcal{X}_n^{\mathbf{A}_0}$-norm. However since $\mathbf{A}_0$ is invertible the norms $\|(\mu - \mathbf{A}_0) \cdot\|_{\mathcal{X}}$ and of $\|\mathbf{A}_0 \cdot\|_{\mathcal{X}}$ are equivalent which means that $\mathcal{X}_n^{\mathbf{A}} = (D(\mathbf{A}_0^n), \|\mathbf{A}_0^n \cdot\|)$.

Proof of item 2 Direct consequence of item 1 as $\mathbf{A}|_{\mathcal{X}_n^{\mathbf{A}}}$ generates a C_0–semigroup (see [17]).

Proof of item 3 See Proposition 3.2 for the expression of $\mathcal{X}_1^{\mathbf{A}_0}$. The fact that there exists a constant $C > 0$ such that $\|\cdot\|_{1,\mathbf{A}_0} \leq C\|\cdot\|_1$ is straightforward. The reverse inequality is a consequence of the continuity of $\mathbf{M}_f$ and $\mathbf{D} = -\frac{\mathbf{A}_0 + \mathbf{M}_f}{a_\infty} = \partial_x$ from $\mathcal{X}_1^{\mathbf{A}_0}$ to $\mathcal{X}$ (see Lemma C.1).

Proof of item 4 We first identify $D(\mathbf{A}_0^2)$. As in the proof of Proposition 3.2, we start from $\psi = \mathbf{R}(\mu, \mathbf{A}_0)\phi$ with $\phi \in D(\mathbf{A}_0)$ and $\Re\mu \geq 0$. We have that $\mathbf{R}(\mu, \mathbf{A}_0)D(\mathbf{A}_0) = D(\mathbf{A}_0^2)$ and we deduce firstly that $\psi(0) = \psi'(0) = 0$. Moreover the proof of Lemma C.1 shows that $a_\infty\|f^2\psi\|_{\mathcal{X}} \leq C\,\|\phi\|_{1,\mathbf{A}_0}$ which implies that $f^2\psi \in \mathcal{X}$. From the definition of ψ, we have that

$$a_\infty\psi' = -f\psi + \phi - \mu\psi \in D(\mathbf{A}_0)$$

which gives $f\psi' \in \mathcal{X}$. Hence, from Assumption 1, we have $(f\psi)' \in \mathcal{X}$ and $\psi'' \in \mathcal{X}$. To sum up we have shown that:

$$D(\mathbf{A}_0^2) \subset \{\phi \in \mathcal{X},\ \phi'' \in \mathcal{X},\ f\phi' \in \mathcal{X},\ f^2\phi \in \mathcal{X},\ \phi(0) = \phi'(0) = 0\}.$$

Reciprocally, let $\psi \in \{\phi \in \mathcal{X},\ \phi'' \in \mathcal{X},\ f\phi' \in \mathcal{X},\ f^2\phi \in \mathcal{X},\ \phi(0) = \phi'(0) = 0\}$ and define $\phi = a_\infty\psi' + f\psi + \mu\psi$: we will show that $\phi \in D(\mathbf{A}_0)$ noting that $\psi = \mathbf{R}(\mu, \mathbf{A}_0)\phi$ by injectivity of $\mu\mathrm{Id} - \mathbf{A}_0$. We first note that $\phi(0) = 0$.

- $\phi \in \mathcal{X}$ because $\psi \in \mathcal{X}$ and $\psi' \in \mathcal{X}$ and

$$\|f\psi\|_{\mathcal{X}} \leq \|f^2\psi\|_{\mathcal{X}} + \int_{\{f\leq 1\}} |\psi| \leq \left\|f^2\psi\right\|_{\mathcal{X}} + \|\psi\|_{\mathcal{X}} < \infty, \tag{32}$$

- $f\phi \in \mathcal{X}$ because $f\psi' \in \mathcal{X}$, $f^2\psi \in \mathcal{X}$ and $f\psi \in \mathcal{X}$ thanks to (32),
- $\phi' \in \mathcal{X}$ because $\psi'' \in \mathcal{X}$, $\psi' \in \mathcal{X}$ (thanks to $f\psi' \in \mathcal{X}$), and $(f\psi)' = f'\psi + f\psi'$ is such that $f\psi' \in \mathcal{X}$ and

$$\|f'\psi\|_{\mathcal{X}} \overset{Assumption\ 1}{\leq} \int_{\{x\leq 1\}} f'|\psi| + c\|f\psi\|_{\mathcal{X}} \lesssim \|f\psi\|_{\mathcal{X}} + \|\psi\|_{\mathcal{X}} < \infty. \tag{33}$$

Hence $\phi \in D(\mathbf{A}_0)$ which gives $\psi = \mathbf{R}(\mu, \mathbf{A}_0)\phi \in D(\mathbf{A}_0^2)$ and it follows that $D(\mathbf{A}_0^2) = \{\phi \in \mathcal{X},\ \phi'' \in \mathcal{X},\ f\phi' \in \mathcal{X},\ f^2\phi \in \mathcal{X},\ \phi(0) = 0,\ \phi'(0) = 0\}$. As for the norms, for all $\phi \in \mathcal{X}_2^{\mathbf{A}_0}$

$$\begin{aligned}
\|\mathbf{A}_0^2\phi\|_{\mathcal{X}} &\lesssim \left(\|\phi''\|_{\mathcal{X}} + \|f\phi'\|_{\mathcal{X}} + \|f'\phi\|_{\mathcal{X}} + \|f^2\phi\|_{\mathcal{X}}\right) \\
&\overset{(33)}{\lesssim} \left(\|\phi''\|_{\mathcal{X}} + \|f\phi\|_{\mathcal{X}} + \|f\phi'\|_{\mathcal{X}} + \|f^2\phi\|_{\mathcal{X}} + \|\phi\|_{\mathcal{X}}\right) \\
&\overset{(32)}{\lesssim} \left(\|\phi''\|_{\mathcal{X}} + \|\phi\|_{\mathcal{X}} + \|f\phi'\|_{\mathcal{X}} + \|f^2\phi\|_{\mathcal{X}}\right) = \|\phi\|_2.
\end{aligned}$$

For the reverse inequality $\|\phi\|_2 \lesssim \|\phi\|_{2,\mathbf{A}_0}$, only the terms $\|f\phi'\|_{\mathcal{X}}$ and $\|\phi''\|_{\mathcal{X}}$ require additional attention. From the continuity of $\mathbf{M}_f\mathbf{A}_0^{-1} \in \mathcal{L}(\mathcal{X})$ (see (8)), we have

$$\left\|\mathbf{D}^2\mathbf{A}_0^{-2}\phi\right\|_{\mathcal{X}} = \frac{1}{a_\infty^2}\left\|(Id + \mathbf{M}_f\mathbf{A}_0^{-1})^2\phi\right\|_{\mathcal{X}} \lesssim \|\phi\|_{\mathcal{X}}$$

which gives $\|\phi''\|_{\mathcal{X}} \lesssim \|\phi\|_{2,\mathbf{A}_0}$. We also have $\|f\phi'\|_{\mathcal{X}} = \frac{1}{a_\infty}\|\mathbf{M}_f(\mathbf{A}_0 + \mathbf{M}_f)\phi\|_{\mathcal{X}} \overset{Lemma\ C.1}{\lesssim} \|\phi\|_{2,\mathbf{A}_0}$ which concludes the proof.

Proof of item 5 The proof is essentially the same as the one of the previous items. The domain of $D(\mathbf{C}_\alpha^2)$ is the same as $D(\mathbf{A}_0^2)$. Up to scaling f, the two previous items show that $\|\cdot\|_{1,\alpha}$ (resp. $\|\cdot\|_{2,\alpha}$) is equivalent to $\|\cdot\|_1$ (resp. $\|\cdot\|_2$) hence the different norms $\|\cdot\|_{1,\alpha}$ for $\alpha > 0$ are equivalent, the same is true for $\|\cdot\|_{2,\alpha}$.

E Lemmas for the continuity of $\mathbf{U}_a$ and $\mathbf{V}_a$

Lemma E.1 *If Assumption 1 is satisfied, then for all $C \geq 0$ and $\underline{a} > 0$, there are two constants $t_0 > 0$ and $C' > 0$ such that for all $\varphi \in \mathcal{X}$*

$$\forall u \geq t_0, \quad e^{C\cdot}\mathbf{U}_{\underline{a}}(u)|\varphi| \leq e^{-C'u^2}|\varphi(\cdot - u)|H(\cdot - u) \quad a.s., \tag{34a}$$

$$\left\|e^{C\cdot}\int_{t_0}^{\infty}\mathbf{U}_{\underline{a}}(u)|\varphi|\right\|_{\mathcal{X}} \leq C'\|\varphi\|_{\mathcal{X}}. \tag{34b}$$

Proof. Let us first bound the following function:

$$\begin{aligned}
\forall x \geq 0, \forall u \geq 0, \quad |e^{Cx}\mathbf{U}_{\underline{a}}(u)\varphi|(x) &= e^{C(x) - \frac{1}{\underline{a}}\int_{x-u}^{x} f dv}|\varphi|(x-u)H(x-u) \\
&\overset{y=x-u\geq 0}{=} e^{C(y+u) - \frac{1}{\underline{a}}\int_0^u f(v+y)dv}|\varphi|(y)H(y) \\
&\overset{\text{Rem. } 1}{\leq} e^{C(y+u) - \frac{1}{\underline{a}}\int_0^u (f(v)+f(y))dv}|\varphi|(y)H(y) \\
&\overset{Assumption\ 1\ (iv)}{\leq} e^{C(y+u) - \frac{1}{\underline{a}}\left(f(y)u + \int_0^1 f(v)dv + c\int_1^u v dv\right)}|\varphi|(y)H(y) \\
&= e^{C(y+u) - \frac{c}{2\underline{a}}u^2 - \frac{u}{\underline{a}}f(y) + c'}|\varphi|(y)H(y)
\end{aligned}$$

with $c' = \int_0^1 f(v)dv$. Let be $h_u(y) = C(y+u) - \frac{c}{2\underline{a}}u^2 - \frac{u}{\underline{a}}f(y) + c'$. For $y \geq 1$, we have by Rem. 1, $h_u(y) \leq C(y+u) - \frac{c}{2\underline{a}}u^2 - c\frac{u}{\underline{a}}y + c' = c' + x(C - c\frac{u}{\underline{a}}) - \frac{c}{2\underline{a}}u^2$. For $u \geq \frac{C\underline{a}}{c}$, we find $h_u(y) \leq c' - \frac{c}{2\underline{a}}u^2$. This gives $h_u(y) \leq -C_1u^2$ for u large enough for a new constant $C_1 > 0$. For $y \leq 1$, we have $h_u(y) \leq C(1+u) - \frac{c}{2\underline{a}}u^2 \leq -C_2u^2$ for u large enough for a new constant $C_2 > 0$. Hence, we found that there is a constant $\tilde{C} > 0$ such that

$$\exists t_0 > 0, \forall y \geq 0, \forall u \geq t_0,\ h_u(y) \leq -\tilde{C}u^2.$$

This implies that $e^{C\cdot}\mathbf{U}_{\underline{a}}(u)|\varphi| \leq e^{-\tilde{C}u^2}|\varphi(. - u)|H(. - u)$ for $u \geq t_0$ and gives the first inequality of the lemma. It also gives the second inequality.

Lemma E.2 *Grant Assumption 1 for (35a) or 2 for (35b). There is a constant $C > 0$ such that $\forall\varphi \in \mathcal{X}_2^{\mathbf{A}}$, $\forall a \in C^0(\mathbb{R}^+)$ and $\forall t, u \geq 0$:*

$$\|\mathbf{U}_a(t+u,t)\varphi\|_{2,\mathbf{A}} \leq C\|\varphi\|_{2,\mathbf{A}} \tag{35a}$$
$$\|\mathbf{V}_a(t+u,t)\varphi\|_{2,\mathbf{A}} \leq C(\|\rho(a)\|_\infty + \|\varphi\|_{2,\mathbf{A}}). \tag{35b}$$

In particular, these operators leave $\mathcal{X}_2^{\mathbf{A}}$ invariant.

Proof.
We start with the simpler case of $\mathbf{U}_a$. We first show that $\mathbf{U}_a(t+u,t)\varphi$ belongs to $W^{2,1}_{loc}(\mathbb{R}_+)$ if $\varphi \in \mathcal{X}_2^{\mathbf{A}}$. For $\varphi \in \mathcal{X}_2^{\mathbf{A}}$ and $t \geq s \geq 0$, we write $u(t,\cdot) = \mathbf{U}_a(t,s)\varphi$. We note that $u(t,x) = q(t,s,x)(\mathbf{T}_r(t-s)\varphi)(x)$ where $(\mathbf{T}_r(t))_{t\geq 0}$ is the C^0-semigroup of right translations and $x \to q(t,s,x) \in C^2(\mathbb{R}_+)$ is a bounded function with bounded derivatives. It is known that $\mathbf{T}_r(t)$ leaves $\{\varphi \in W^{2,1}(\mathbb{R}_+),\ \varphi(0) = \varphi'(0) = 0\}$ invariant[§]. It follows that $u(t,\cdot) \in W^{2,1}_{loc}(\mathbb{R}_+)$. We can thus take the derivatives of $\mathbf{U}_a(t+u,t)\varphi$ in order to compute norms.

Let us now bound almost everywhere $\mathbf{U}_a(t+u,t)\varphi$, $f^2\mathbf{U}_a(t+u,t)\varphi$, $f(\mathbf{U}_a(t+u,t)\varphi)'$ and $(\mathbf{U}_a(t+u,t)\varphi)''$ in order to show that $\|\mathbf{U}_a(t+u,t)\varphi\|_2 \lesssim \|\varphi\|_2$. In particular, this will show that these functions are integrable. We then note from Assumption 1 that there is $C > 0$ such that $f(x) \leq C\exp(Cx)$ for all $x \geq 0$.

- Let $k \in \{0,2\}$, from Lemma E.1, there are constants $C > 0$ and $t_0 > 0$ such that

$$f^k|\mathbf{U}_a(t+u,t)\varphi| \leq \begin{cases} f^k\mathbf{T}_r(u)|\varphi|, & \text{if } 0 \leq u \leq t_0 \\ e^{-Cu^2}\mathbf{T}_r(u)|\varphi| & \text{otherwise} \end{cases}$$

  which gives for some new constant C independent of a

$$\left\|f^k\mathbf{U}_a(t+u,t)\varphi\right\|_{\mathcal{X}} \leq \begin{cases} C(\|\varphi\|_{\mathcal{X}} + \left\|f^2\varphi\right\|_{\mathcal{X}}), & \text{if } 0 \leq u \leq t_0 \\ C\,\|\varphi\|_{\mathcal{X}} & \text{otherwise.} \end{cases}$$

  that is there is $C > 0$ such that for all $t, u \geq 0$, $\left\|f^k\mathbf{U}_a(t+u,t)\varphi\right\|_{\mathcal{X}} \leq C\,\|\phi\|_2$.

- The derivative $(\mathbf{U}_a(t+u,t)\varphi)'$ is bounded by

$$\begin{aligned}|(\mathbf{U}_a(t+u,t)\varphi)'| &\lesssim X_u'\mathbf{U}_{\bar{a}}(u)|\varphi| + \mathbf{U}_{\bar{a}}(u)|\varphi'| \\ &\leq C\left[(u + X_u)\mathbf{U}_{\bar{a}}(u)|\varphi| + \mathbf{U}_{\bar{a}}(u)|\varphi'|\right]\end{aligned} \tag{36}$$

  for $C \geq 1$ where $X_u(x) \overset{def}{=} \int_0^u f(v+x-u)dv$ and $X_u'(x) = \int_0^u f'(v+x-u)dv \overset{Assumption\ 1}{\leq} C(u + X_u(x))$. Using the boundedness of $x \to xe^{-x}$, we find that

$$|(\mathbf{U}_a(t+u,t)\varphi)'| \leq C\left[|\phi| + u\mathbf{U}_{\bar{a}}(u)|\varphi| + \mathbf{U}_{\bar{a}}(u)|\varphi'|\right].$$

[§]It is the domain of the square of its infinitesimal generator

The only remaining term to study is:

$$u f \mathbf{U}_{\bar a}(u)|\varphi| \overset{Lemma\ E.1}{\leq} \begin{cases} Cf\mathbf{T}_r(u)|\varphi|, & \text{if } 0 \leq u \leq t_0 \\ C\mathbf{T}_r(u)|\varphi| & \text{otherwise} \end{cases}$$

which implies that there is a constant $C > 0$ such that for all $u, t \geq 0$

$$\|f(\mathbf{U}_a(t+u,t)\varphi)'\|_{\mathcal{X}} \leq C \|\varphi\|_2 .$$

Similarly, using that $f'' \overset{Assumption\ 1}{\leq} C(1+f)$ to get $X_u'' \leq C(u + X_u)$ for some C, we find $\forall x, u \geq 0$

$$\begin{aligned} |(\mathbf{U}_a(t+u,t)\varphi)''| &\lesssim ((X_u'' + X_u'^2)\mathbf{U}_{\bar a}(u)|\varphi| + 2X_u'\mathbf{U}_{\bar a}(u)|\varphi'| + \mathbf{U}_{\bar a}(u)|\varphi''|) \\ &\leq C\left[(u + u^2 + X_u + 2uX_u + X_u^2)\mathbf{U}_{\bar a}(u)|\varphi| + 2(u + X_u)\mathbf{U}_{\bar a}(u)|\varphi'| + \mathbf{U}_{\bar a}(u)|\varphi''|\right] . \end{aligned} \tag{37}$$

As above, using Lemma E.1 and the boundedness of $x \to x^2 e^{-x}$, there is a constant $C > 0$ such that for all $u, t \geq 0$

$$\|(\mathbf{U}_a(t+u,t)\varphi)''\|_{\mathcal{X}} \leq C \|\varphi\|_2 .$$

Putting all of this together, this shows that there is $C > 0$ independent of a such that for all $u, t \geq 0$:

$$\varphi \in \mathcal{X}_2^{\mathbf{A}}, \ \|\mathbf{U}_a(t+u,t)\varphi\|_2 \leq C \|\varphi\|_2 .$$

Using Lemma 3.1, we then get $\|\mathbf{U}_a(t+u,t)\varphi\|_{2,\mathbf{A}} \leq C \|\varphi\|_{2,\mathbf{A}}$. We now look at $\mathbf{V}_a$ by taking advantage of the above computations. For $k \in \{0, 2\}$:

$$\left\|f^k \mathbf{V}_a(t+u,t)\varphi\right\|_{\mathcal{X}} \lesssim \|\varphi\|_{2,\mathbf{A}} + \frac{\|\rho(a)\|_\infty}{a_\infty(a_\infty - \eta)} \|f^k \int_t^{t+u} \mathbf{U}_{\bar a}(t+u-r)(fg_\infty)dr\|_{\mathcal{X}}$$

The integral term is bounded by $\|f^k \int_0^\infty \mathbf{U}_{\bar a}(r)(fg_\infty)dr\|_{\mathcal{X}} \overset{Lemma\ E.1}{<} \infty$ leading to:

$$\left\|f^k \mathbf{V}_a(t+u,t)\varphi\right\|_{\mathcal{X}} \lesssim (\|\rho(a)\|_\infty + \|\varphi\|_{2,\mathbf{A}}).$$

We now look at the case of $f(\mathbf{V}_a(t+u,t)\varphi)'$, only the integral term requires additional analysis. We have

$$|(\mathbf{U}_a(t+u,r)(fg_\infty))'| \lesssim [(X_{t+u-r} + (u+t-r))\mathbf{U}_{\bar a}(u+t-r)(fg_\infty) + \mathbf{U}_{\bar a}(u+t-r)|(fg_\infty)'|)]$$

which is bounded in every neighborhood of x thanks to the particular shape of $U_{\bar a}$. We can thus apply Lebesgue's dominated convergence to differentiate under the integral sum to get:

$$\begin{aligned} \|f\partial_x \int_t^{t+u} \mathbf{U}_a(t+u,r)(fg_\infty))\|_{\mathcal{X}} &= \| \int_t^{t+u} f\partial_x \mathbf{U}_a(t+u,r)(fg_\infty))\|_{\mathcal{X}} \\ &\overset{(36)}{\lesssim} \left\|\int_0^u fX_r U_{\bar a}(r)(fg_\infty)dr\right\|_{\mathcal{X}} + \left\|\int_0^u fU_{\bar a}(r)|(fg_\infty)'|dr\right\|_{\mathcal{X}} + \left\|\int_0^u rfU_{\bar a}(r)(fg_\infty)dr\right\|_{\mathcal{X}} =_u O(1). \end{aligned}$$

Indeed, the only non-trivial inequality in the above expression comes from the first integral term. From Assumption 1, we have $f(x)X_u(x) \leq Ce^{2Cx+Cu}$ for some constant $C > 0$ and the rest follows from Lemma E.1. Similarly

$$\begin{aligned} \|\partial_x^2 \int_t^{t+u} \mathbf{U}_a(t+u,r)(fg_\infty))\|_{\mathcal{X}} &= \| \int_t^{t+u} \partial_x^2 \mathbf{U}_a(t+u,r)(fg_\infty))\|_{\mathcal{X}} \\ &\overset{(37)}{\lesssim} \left\|\int_0^u U_{\bar a}(r)|(fg_\infty)''|dr\right\|_{\mathcal{X}} + \left\|\int_0^u (r + X_r)U_{\bar a}(r)|(fg_\infty)'|dr\right\|_{\mathcal{X}} + \\ &\qquad \left\|\int_0^u (r + r^2 + X_r + 2rX_r + X_r^2)U_{\bar a}(r)(fg_\infty)dr\right\|_{\mathcal{X}} =_u O(1). \end{aligned}$$

This shows that there is a constant $C > 0$ independent of a such that for all $t, u \geq 0$, $\forall \phi \in \mathcal{X}_2^{\mathbf{A}_0}$

$$\|f(\mathbf{V}_a(t+u,t)\varphi)'\|_{\mathcal{X}}\,,\ \|(\mathbf{V}_a(t+u,t)\varphi)''\|_{\mathcal{X}} \leq C(\|\rho(a)\|_\infty + \|\varphi\|_2).$$

or

$$\|\mathbf{V}_a(t+u,t)\varphi\|_2 \leq C\left[\|\rho(a)\|_\infty + \|\varphi\|_2\right].$$

We conclude as for the case of $\mathbf{U}_a$.

Proposition E.1 *Grant Assumption 1. For all $\varphi \in \mathcal{X}_2^{\mathbf{A}}$, $\forall t \geq s \geq 0$, the mapping $a \to \mathbf{U}_a(t,s)\varphi$ is C^1 from $C^0([t,s])$ into $\mathcal{X}_2^{\mathbf{A}}$ and*

$$d\left[\mathbf{U}_a(t,s)\varphi\right]\cdot b = \left(\int_s^t \frac{f(v+.-t)b(v)}{(a_\infty+\rho(a(v)))^2}\rho'(a(v))dv\right)\mathbf{U}_a(t,s)\varphi.$$

Additionally, grant Assumption 2, then the mapping $a \to \mathbf{V}_a(t,s)\varphi$ is C^1 from $C^0([t,s])$ into $\mathcal{X}_2^{\mathbf{A}}$.

Proof. We consider $\phi \in \mathcal{X}_2^{\mathbf{A}}$ and $a \to \mathbf{U}_a(t,s)\phi$, the case of $\mathbf{V}_a(t,s)$ is similar. Recall from (14) that we write $\widetilde{a}(t) \stackrel{def}{=} a_\infty + \rho(a(t))$. The mapping $a \to \widetilde{a}$ being C^1 from $C^0([t,s])$ into itself, it is enough to prove the differentiability of $\mathbf{F}: a \to \exp\left(-\int_s^t f(v+\cdot-t)a(v)dv\right)\mathbf{T}_r(t-s)\phi$ from $C^0([t,s])$ into $\mathcal{X}_2^{\mathbf{A}}$ at any point a such that $\underline{a} \stackrel{def}{=} \min a > 0$. We thus consider such a point $a \in C^0([s,t])$. It is convenient to define the following functions $\Delta_{t,s} \stackrel{\text{def}}{=} \mathbf{F}(a+b) - \mathbf{F}(a) - d\mathbf{F}(a)\cdot b = \mathcal{E}_{t,s}\mathbf{T}_r(t-s)\varphi$ where

$$d\mathbf{F}(a)\cdot b \stackrel{def}{=} e^{-X_{t,s}(a)}X_{t,s}(b)\mathbf{T}_r(t-s)\varphi,\quad \mathcal{E}_{t,s} \stackrel{def}{=} e^{-X_{t,s}(a+b)} - e^{-X_{t,s}(a)} + e^{-X_{t,s}(a)}X_{t,s}(b)$$

and $X_{t,s}(a) \stackrel{def}{=} x \to \int_s^t f(v+x-t)a(v)dv$. Using the Taylor formula with integral reminder, one finds $\forall b \in B_{C^0([s,t])}(0,\delta)$

$$\mathcal{E}_{t,s} = e^{-X_{t,s}(a+b)}X_{t,s}(b)^2\int_0^1 e^{uX_{t,s}(b)}udu.$$

By Assumption 1, there is a constant $C > 0$ such that for all $u \geq 0$, $X_{0,u}(1)', X_{0,u}(1)'' \leq C(u + X_{0,u}(1))$. Hence, using the monotony properties of f, we find that for $k \in \{0,1,2\}$

$$|\mathcal{E}_{t,s}^{(k)}| \leq P_k(X_{0,t-s}(1), t-s, \delta, \underline{a})e^{-(\underline{a}-2\delta)X_{0,t-s}(1)}\|b\|^2_{C^0([s,t])},\quad a.s. \tag{38}$$

for polynomials $P_k(\cdot, t-s, \delta, \underline{a}) \in \mathbb{R}_{k+2}[X]$ with positive coefficients. Differentiating $\Delta_{t,s}$, we find

$$\begin{aligned}(\Delta_{t,s})' &= \mathcal{E}'_{t,s}\mathbf{T}_r(t-s)\varphi + \mathcal{E}_{t,s}\mathbf{T}_r(t-s)\varphi' \\ (\Delta_{t,s})'' &= \mathcal{E}''_{t,s}\mathbf{T}_r(t-s)\varphi + 2\mathcal{E}'_{t,s}\mathbf{T}_r(t-s)\varphi' + \mathcal{E}_{t,s}\mathbf{T}_r(t-s)\varphi''.\end{aligned}$$

We now use Lemma E.2 and estimate (38) in the case $\delta < \underline{a}/2$ to show that

$$\varphi \in \mathcal{X}_2^{\mathbf{A}},\ \|\Delta_{t,s}(a)\|_2 \lesssim \|\varphi\|_2\,\|b\|^2_{C^0([s,t])}.$$

The fact that $d\mathbf{F}(a)$ is a continuous linear mapping is straightforward and thus we obtain that $\mathbf{F}$ is differentiable at a. This shows that $a \to \mathbf{U}_a(t,s)\varphi$ is C^1 from $C^0([s,t])$ into $\mathcal{X}_2^{\mathbf{A}}$.

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